LECTURE

MODELLING THE MARINE ENVIRONMENT

Lecturer: Prof. Nguyen Ky Phung

MSc. Dang Thi Thanh Le

Co-funded by the Erasmus+ Programme of the European Union

Lecture 7

PROCESS OF SUBSTANCE TRANSMISSION

Lecturer: Prof. Nguyen Ky Phung
MSc. Dang Thi Thanh Le

Co-funded by the
Erasmus+ Programme
of the European Union

Lecture 7A

MASS BALANCING EQUATION

Cán Bộ Giảng Dạy: GS.TS. Nguyễn Kỳ Phùng
ThS. Đặng Thị Thanh Lê

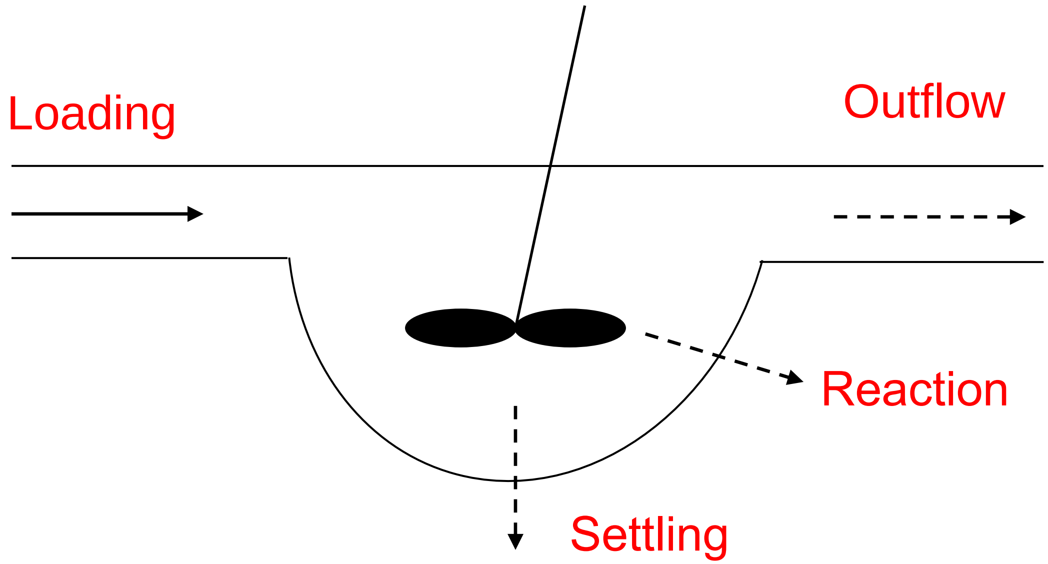
Co-funded by the
Erasmus+ Programme
of the European Union

CONTENTS

- **MASS balancing equation**
- **Steady - state system Solution**
  - Transfer functions
  - Residence
- **Unsteady - state system Solution**
  - Impulse loading
  - Step loading
  - Linear loading
  - Exponential loading
  - Sinusodial loading

MASS BALANCE FOR A WELL-MIXED LAKE

A completely mixing system. (CSTR – Continuously Stirred Tank Reactor) It's the simplest system that can be used to model a real water body.



For a finite time period, the mass balance of the system is described by the equation:

Accumulation = Loading - OutFlow - Reaction - Settling

Co-funded by the Erasmus+ Programme of the European Union

MASS BALANCE FOR A WELL-MIXED LAKE (Cont)

❖ **Accumulation:** represent the change of Mass M of the system over a period of time t.

$$\left.\begin{aligned} \text{Accumulation} &= \frac{\Delta M}{\Delta t} \\ c = \frac{M}{V} &\rightarrow M = Vc \end{aligned}\right\} \rightarrow \text{Accumulation} = \frac{\Delta Vc}{\Delta t} \quad [MT^{-1}]$$

If V is constant

$$\text{Accumulation} = V\frac{\Delta c}{\Delta t} \rightarrow V\frac{dc}{dt}$$

$$\frac{dc}{dt} > 0 \rightarrow \text{Increased accumulation}$$

$$\frac{dc}{dt} < 0 \rightarrow \text{Reduced accumulation}$$

$$\frac{dc}{dt} = 0 \rightarrow \text{Constant}$$

MASS BALANCE FOR A WELL-MIXED LAKE (cont)

❖ Loading :

$$\text{Loading} = W(t) = Qc_{in}(t) \quad [MT^{-1}]$$

Q : Flow$[L^3T^{-1}]$

$c_{in}(t)$: concentration$[ML^{-3}]$

❖ **Point source** (source identity, convenient measurement, continuality)

+ Municipal wastewater
+ Industrial wastewater
+ Tributary

❖ **Distribution source**

- Agriculture
- Atmosphere
- Water flowing from the city.
- Groundwater

Co-funded by the
Erasmus+ Programme
of the European Union

MASS BALANCE FOR A WELL-MIXED LAKE (cont)

❖ Outflow:

❖ **A WELL-MIXED LAKE** : $c = c_{out}$

$$\rightarrow \text{Outflow} = Qc_{out} = Qc$$

❖ Reaction:

$$\text{Reaction} = kM = kVc$$

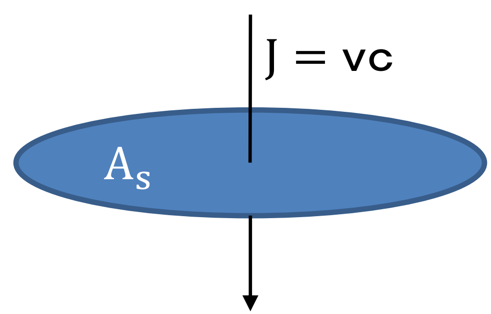
k: First order reaction coefficient [T^{-1}]

❖ Settling:

$$\text{Settling} = vA_s c$$

$$V = A_s H$$

$$\rightarrow \text{Settling} = \frac{v}{H} Vc = k_s Vc$$



v: apparent settling velocity[LT^{-1}]

k_s: a first – oder settling rate constant= v/H [T^{-1}]

H: depth[L]

Co-funded by the Erasmus+ Programme of the European Union

MASS BALANCE FOR A WELL-MIXED LAKE (cont)

❖ Mass balancing equation

$$V\frac{dc}{dt} = W(t) - Qc - kVc - vA_s c$$

- Dependent variable: c
- Independent variable : t
- Impact function: W(t), which represents external influence on the system
- Parameter: V, Q, k, v, và A_s

Co-funded by the Erasmus+ Programme of the European Union

MASS BALANCE FOR A WELL-MIXED LAKE (cont)

- **Mass balance equation**

$$V\frac{dc}{dt} = W(t) - Qc - kVc - vA_s c = 0$$

- **Solution**

$$\mathrm{c} = \frac{\mathrm{W}}{\mathrm{Q+kV+vA_s}} \text{ hoặc } \mathrm{c} = \frac{\mathrm{W}}{\mathrm{a}}$$

- **Assimilation agent (self-cleansing ability)**

$$\mathrm{a} = \mathrm{Q} + \mathrm{kV} + \mathrm{vA_s}$$

STEADY - SYSTEM SOLUTION(cont)

Example 1:

A lake has the following characteristics: volume = 50,000 m^3, depth = 2m, inflow = outflow = 7500m^3/day, temperature = 25^0C. The lake receives the input of pollutants from three sources:

- a factory discharge of 50 kg/day
- a flux from the air with a load of 0.6 g/m^2/day
- the inflow stream has a concentration of 10mg/l

If the pollutant decay at the rate of 0.25/day at 20^0C (θ=1.05).

a) Calculation of assimilation agents.

b) Determine the steady – state concentration

c) Calculates the mass per time for each term in the mass balance and displays your results on a plot.

STEADY - SYSTEM SOLUTION(cont)

❖Transfer function

$$c = \frac{W}{Q + kV + vA_s} = \frac{Qc_{in}}{Q + kV + vA_s}$$

$$\rightarrow \frac{c}{c_{in}} = \beta = \frac{Q}{Q + kV + vA_s} : \text{Transfer function}$$

- Transfer function : it specified how the system input is transformed into an output
- $\beta \ll 1$: the mechanical cleaning of the lake will reduce the concentration of pollution (high assimilation capacity)
- $\beta \rightarrow 1$: weak lake cleaning mechanics

STEADY - SYSTEM SOLUTION(cont)

❖Residence time

❖ The residence time of a substance E represents the mean amount of time that a particle of E would reside in a system.

$$\tau_E = \frac{E}{|dE/dt|_{\pm}}$$

E: Quantity of E in a specified volume [M or ML^{-3}]

$|dE/dt|_{\pm}$: absolute value of source or lake [MT^{-1}or $ML^{-3}T^{-1}$]

- The residence time of the water in the lake: $\tau = \frac{V}{Q}$
- The residence time of pollutants in the lake

$$\tau = \frac{Vc}{Qc + kVc + vA_s c} = \frac{V}{Q + kV + vA_s}$$

Co-funded by the
Erasmus+ Programme
of the European Union

STEADY - SYSTEM SOLUTION(cont)

Example 2:

A lake has the following characteristics: volume = 50,000 m^3, depth = 2m, inflow = outflow = 7500m^3/day, temperature = 25^0C. The lake receives the input of pollutants from three sources:

- a factory discharge of 50 kg/day
- a flux from the air with a load of 0.6 g/m^2/day
- the inflow stream has a concentration of 10mg/l

If the pollutant decay at the rate of 0.25/day at 20^0C (θ=1.05).

a) Inflow concentration.

b) Transfer function

c) The residence time of the water

d) The residence time of pollutants

STEADY - SYSTEM SOLUTION(cont)

Exercises: A lake has the following characteristics.:

Surface area = $2x10^5$ m^2, Medium depth = 3 m, Q in= Q out = 45000 m^3/day, inflow BOD concentration = 4mg/l, residence time = 2 week

A subdivision housing 1 000 people will discharges raw sewage into this system. Individual contributes about

150 x 3.785 litter/day

and 0.25 x 453.6 g/day.

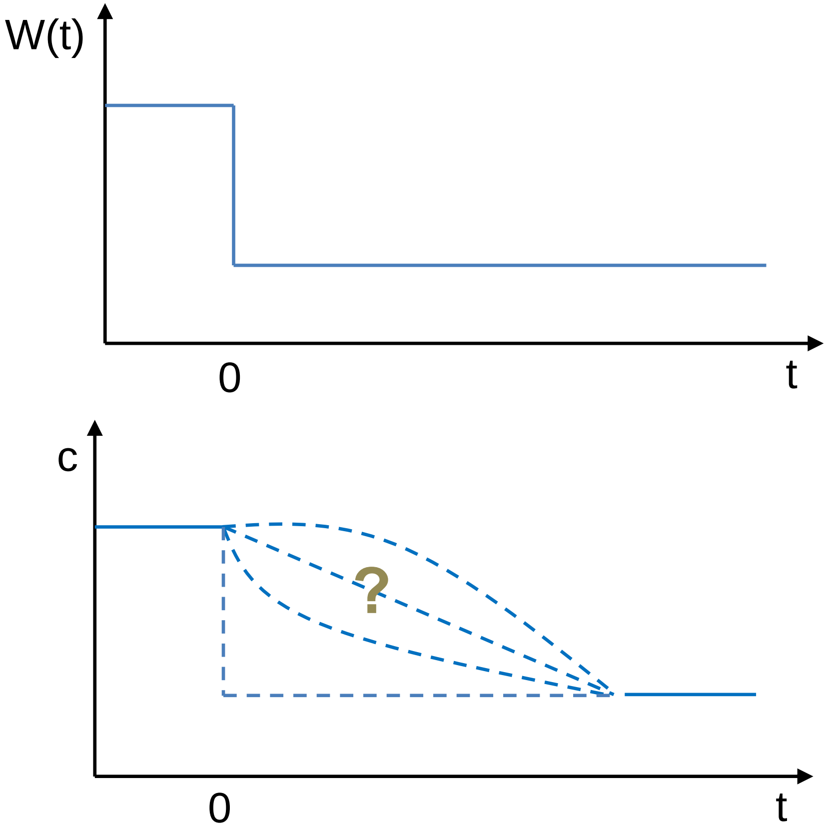
a) Determine the BOD concentration of wastewater (mg/l).

b) If BOD decay = 0.1 day^{-1} and settles at rate of 0.1 m/day. Calculate the assimilation factor for lake prior to building subdivision housing .

c) Calculation of transfer function factor after subdivision housing.

d) Determine the steady concentration of the lake in both cases with and without subdivision housing.

UNSTEADY- SYSTEMS SOLUTION (CONT)



4 water quality recovery scenarios with reduced pollutant load

At a certain point, if the reduction in pollutant emissions takes place, two questions will be asked as follows:

- How long will the water quality be restored?
- What is the "shape" of this recovery?

UNSTEADY- SYSTEMS SOLUTION (CONT)

Co-funded by the Erasmus+ Programme of the European Union

To determine the path of pollutants, consider the equation of mass balance:

$$V\frac{dc}{dt} = W(t) - Qc - kVc - vA_s c$$

$$\leftrightarrow \frac{dc}{dt} = \frac{W(t)}{V} - \frac{Q}{V}c - kc - \frac{v}{H}c$$

$$\leftrightarrow \frac{dc}{dt} + \lambda c = \frac{W(t)}{V} \qquad (*)$$

With $\lambda = \frac{Q}{V} + k + \frac{v}{H}$: eigenvalue

If Q, V, k, v, H is constant, equation (*) are differential equations, first order, linear, and heterogeneous. The solution consists of two parts:

$$c = c_g + c_p$$

c_g: Generalized solution for the case W(t) = 0

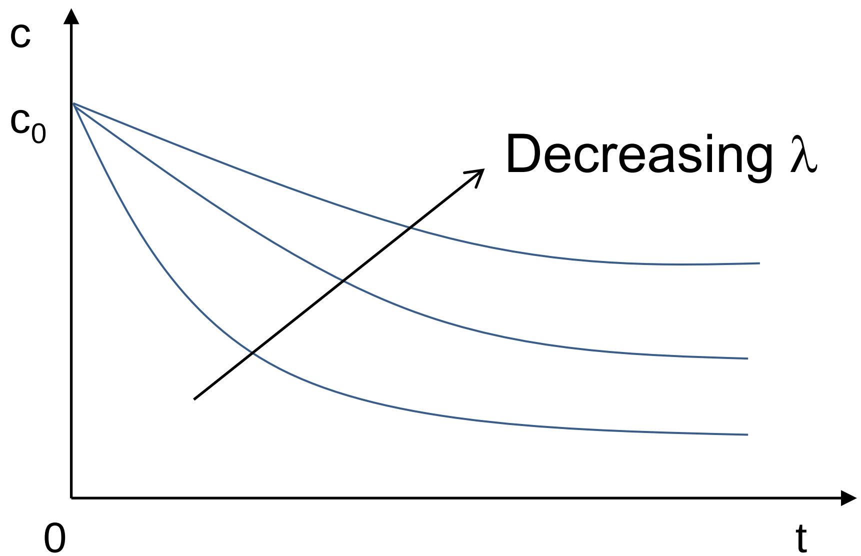
c_p: particular solution for specified form W(t)

UNSTEADY- SYSTEMS SOLUTION (TT)

Co-funded by the Erasmus+ Programme of the European Union

❖General slolution(c_g)

- $W(t) = 0$: $c = c_g$
- At $t = 0$: $c = c_0 \rightarrow$ solution of equation (*) is: $c = c_0 e^{-\lambda t}$



The temporal response of well – mixed lake model following the termination of all loading

UNSTEADY- SYSTEMS SOLUTION (CONT)

❖ Response time

- The time the lake needs to recover..
- t_ϕ = **Response time** ϕ%, which means the time it takes to achieve it. ϕ% the final level of recovery of the lake..

$$t_\phi = \frac{1}{\lambda} \ln \frac{100}{100 - \phi}$$

Response time	t_{50}	$T_{63.2}$	t_{75}	t_{90}	t_{95}	t_{99}
Formula	$0.693/\lambda$	$1/\lambda$	$1.39/\lambda$	$2.3/\lambda$	$3/\lambda$	$4.6/\lambda$

UNSTEADY- SYSTEMS SOLUTION (CONT)

Example 3

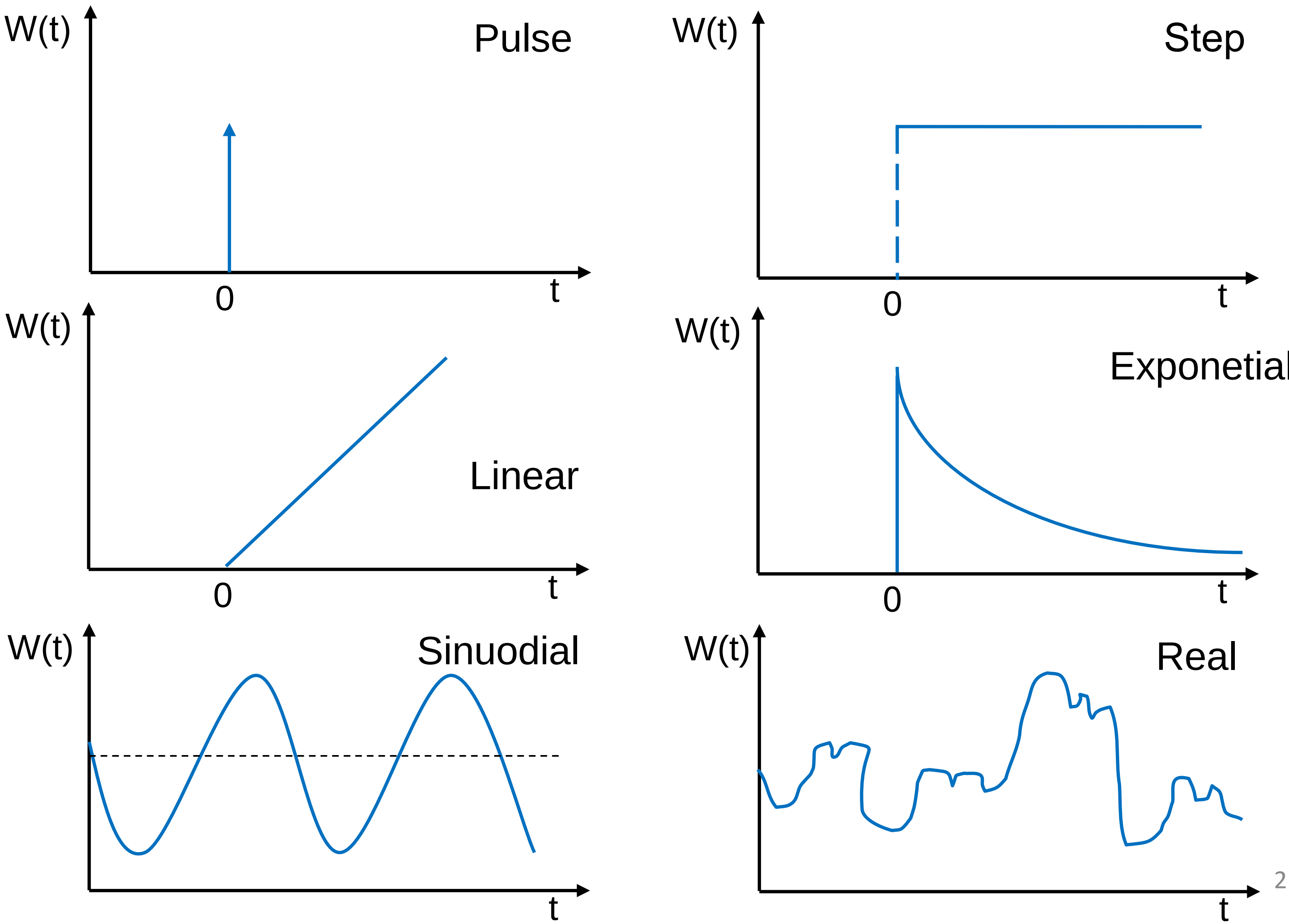
For a lake has the following characteristics**:** volume $=50,000m^3$, depth = 2m, Inflow = outflow = $7500m^3/day$, temperature = 25^0C, waste loading= 140,000g/day, Decay rate= 0.319 d^{-1}

- If the initial concentration is equal to a steady - state level (5.97 mg/l), determine the general solution.
- Determine the response time 75%, 90%, 95%, and 99%

UNSTEADY- SYSTEMS SOLUTION (CONT)

Co-funded by the Erasmus+ Programme of the European Union

❖ **Particular solutions(c_p): Loading function**



UNSTEADY- SYSTEMS SOLUTION (CONT)

❖ **Particular solutions:** Integrating factor method.

$$\frac{dy}{dx} + p(y)x = g(y)$$

$$I = e^{\int p(y)dy}$$

$$\frac{dc}{dt} + \lambda c = \frac{W(t)}{V}$$

$$I = e^{\int \lambda t} = e^{\lambda t}$$

$$e^{\lambda t}\frac{dc}{dt} + e^{\lambda t}\lambda c = e^{\lambda t}\frac{W(t)}{V}$$

$$\frac{d}{dt}\left(e^{\lambda t}c\right) = e^{\lambda t}\frac{dc}{dt} + e^{\lambda t}\lambda c$$

$$\frac{dc}{dt}\left(e^{\lambda t}c\right) = e^{\lambda t}\frac{W(t)}{V}$$

$$e^{\lambda t}c\Big|_0^t = \frac{1}{V}\int e^{\lambda t}W(t)\,dt$$

$$ce^{\lambda t} - c_0 = \frac{1}{V}\int e^{\lambda t}W(t)\,dt$$

$$c(t) = c_0e^{-\lambda t} + \frac{e^{-\lambda t}}{V}\int e^{\lambda t}W(t)\,dt$$

$\mathbf{c_g}$ $\mathbf{c_p}$

UNSTEADY- SYSTEMS SOLUTION (CONT)

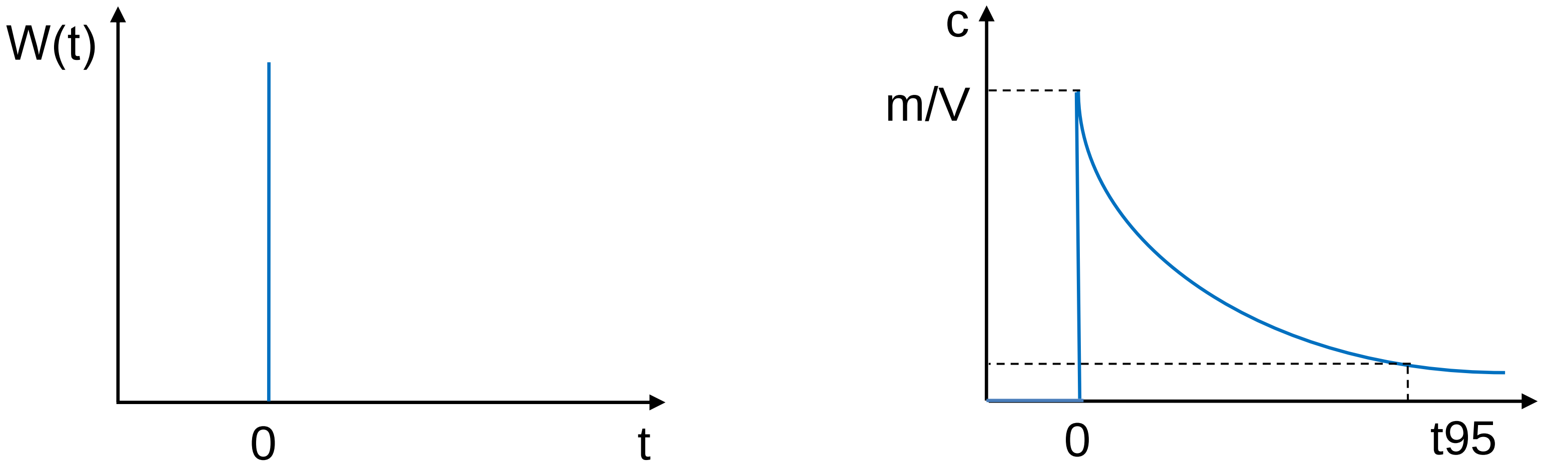
❖ IMPULSE LOADING (SPLII)

Describes the discharge that takes place in a short time.

Impulse function: $W(t) = m\delta(t)$ (m: quantity of pollutant mass[M])

Equation(*): $\frac{dc}{dt} + \lambda c = \frac{m\delta(t)}{V}$

Particular solution: $c = \frac{m}{V} e^{-\lambda t}$



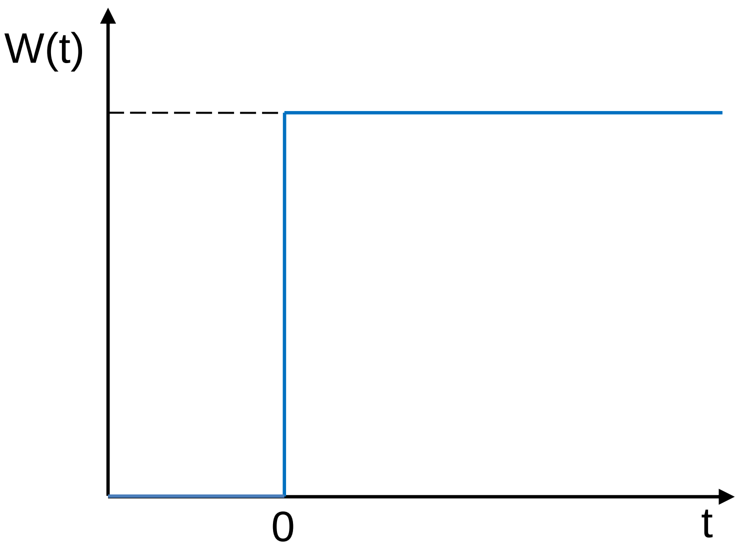
Plot of (a) loading and (b) response for impulse loading

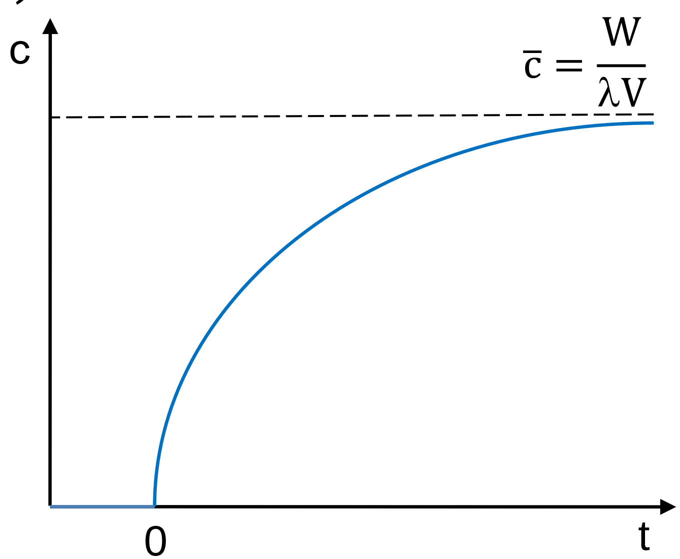
UNSTEADY- SYSTEMS SOLUTION (CONT)

❖ **Step loading (new continuos source)**

❖ Step Funtion: $W(t) = \begin{cases} 0, & t < 0 \\ W, & t \geq 0 \end{cases}$ (W: Loading[MT^{-1}])

Particular solution : $c = \frac{W}{\lambda V}\left(1 - e^{-\lambda t}\right)$





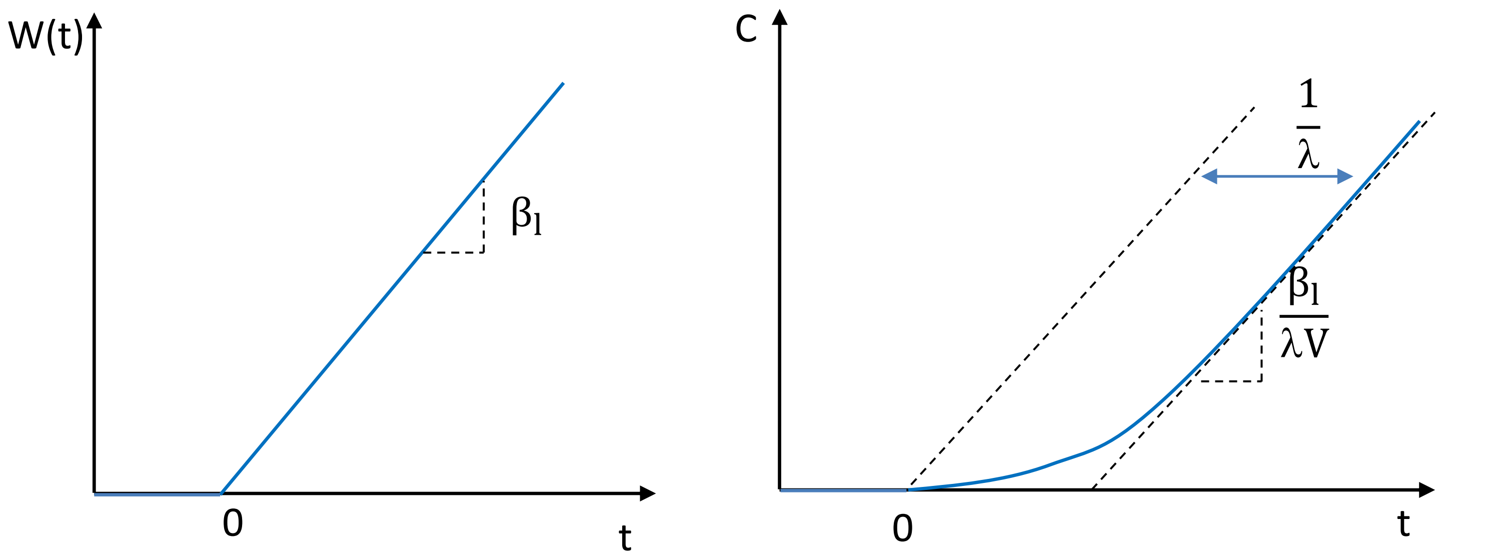
Plot of (a) loading and (b) response for impulse loading

UNSTEADY- SYSTEMS SOLUTION (CONT)

❖ **Linear (Ramp) Loading**

❖ Linear function: $W(t) = \pm\beta_l t$ (β_l: rate of change[MT^{-2}])

Particular solution : $c = \pm\frac{\beta_l}{\lambda^2 V}\left(\lambda t - 1 + e^{-\lambda t}\right)$



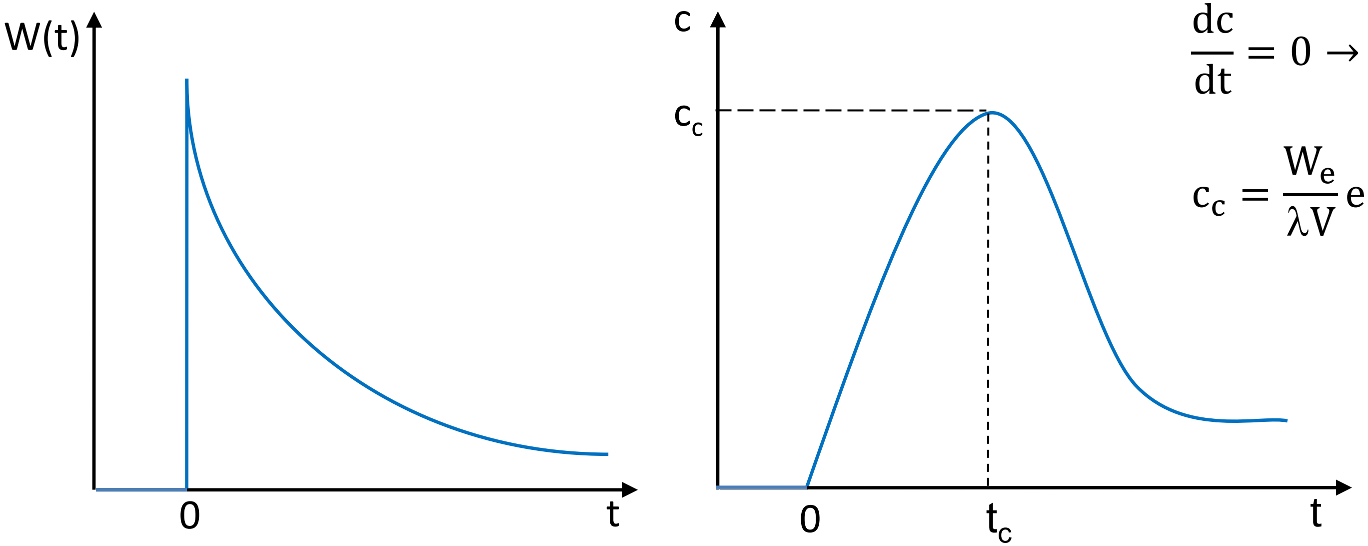
Plot of (a) loading and (b) response for a lineare increasing loading

UNSTEADY- SYSTEMS SOLUTION (CONT)

❖ **Exponential loading**

Exponential function: $W(t) = W_e e^{\pm\beta_e t}$ (W_e: a parameter that denote the value at t = 0 [MT^{-1}], β_e: growth rate (+) or decay (-) of loading [T^{-1}])

Particular solution : $c = \frac{W_e}{V(\lambda \pm \beta_e)}\left(e^{\pm\beta_e t} - e^{-\lambda t}\right)$



$$\frac{dc}{dt} = 0 \rightarrow t_c = \frac{\ln(\beta_e/\lambda)}{\beta_e - \lambda}$$

$$c_c = \frac{W_e}{\lambda V} e^{-\beta_e t_c} = \frac{W_e}{\lambda V}\left(\frac{\beta_e}{\lambda}\right)^{\frac{\beta_e}{\lambda - \beta_e}}$$

Plot of (a) loading and (b) response for an exponential decaying loading

EXERCISES

Exercise 1:

A lake receives two sources of waste of a preserved pollutant from two plants with the following characteristics:

Source of waste from factory 1 with the load of pollutants with the following characteristics:

$$W(t) = \begin{cases} 0 & , x < 1930 \\ 13.2 \times 10^9 (t - 1930) & , 1930 \leq t \leq 1960 \end{cases}$$

Waste source from factory 2 with pollutant load has the following characteristics:

$$W(t) = \begin{cases} 0 & , x < 1900 \\ 229 \times 10^9 e^{0.015(t-1900)} & , 1900 \leq t \leq 1960 \end{cases}$$

where W(t) has units g/year

The initial concentration of the lake is 3mg/l. Characteristics of the lake: outflow 49.1×10^9 m^3/year, V=4880×10^9 m^3. Determine the concentration of pollutants in the lake from 1900 to 1960.

Solution:

Because chloride is a conservative substance.:

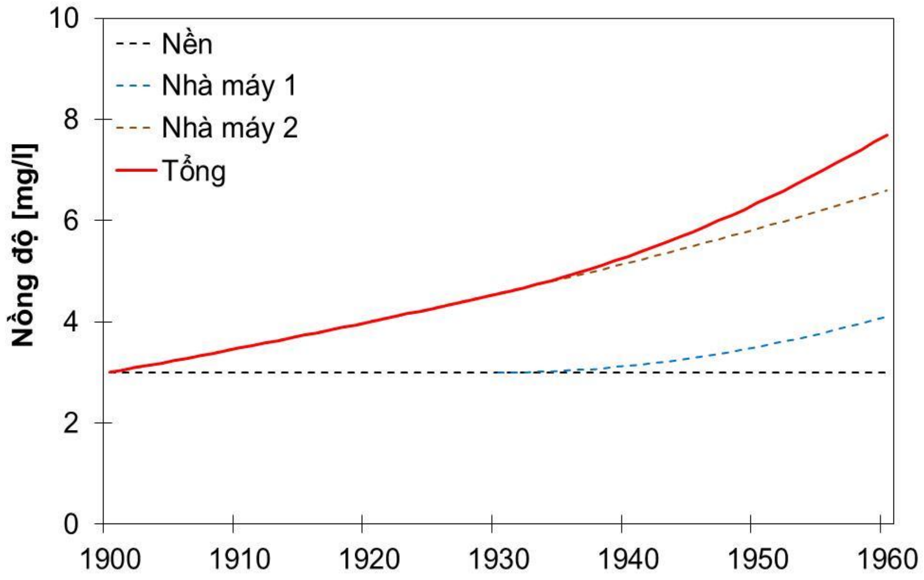
$$\lambda = \frac{Q}{V} = \frac{49.1 \times 10^9}{4880 \times 10^9} = 0.01\ (yr^{-1})$$

From 1900 to 1930

$$c = 3 + \frac{229 \times 10^9}{4880 \times 10^9 (0.01 + 0.015)} \left(e^{0.015(t-1900)} - e^{-0.01(t-1900)}\right)$$

From 1930 to 1960

$$c = 3 + \frac{229\times10^9}{4880\times10^9(0.01+0.015)}\left(e^{0.015(t-1900)} - e^{-0.01(t-1900)}\right) + \frac{13.2\times10^9}{0.01^2\times4880\times10^9}(-1 + e^{-0.01(t-1930)} + 0.01(t-1930))$$



Concentration movement over time

EXERCISES

Exercise 2.

The lake has the following characteristics: Inflow = outflow = 5x10^5 m^3/year, volume = 4x10^7 m^3, surface area = 5x10^6 m^2. The initial concentration of the lake in a stable state is 5μg/l.

The lake receives two sources of waste as follows.:

- In 1994 the lake received a load of 500 kg/year from a fertilizer factory.
- In 1997, the lake began receiving waste from residential areas at a rate of population growth each year. $p = 200e^{0.2t}$ and each citizen will generate phosphorus amounts of 0.5 kg per year.

Calculate the concentration of the lake from 1994 to 2010. Know the deposition speed of phosphorus is 8m /year.