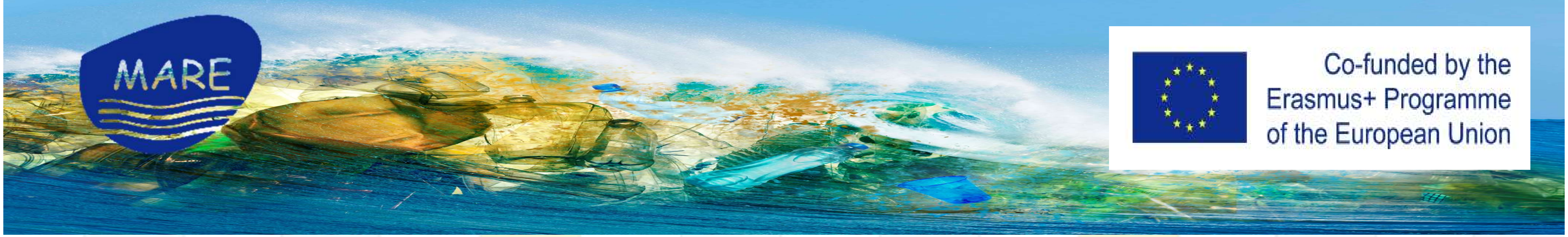

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LECTURE 1

MODELLING AND MATHEMATICAL MODELS

Lecturer: Prof. Nguyen Ky Phung
MSc. Dang Thi Thanh Le
MSc. Tran Thi Kim

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COUSE OUTLINE

1. What is model?
2. What is environmental modelling?
3. Role of environment modelling
4. Types of models
5. Mathematical models?

1. WHAT IS MODEL?

A Model is a simplification of reality that is constructed to gain insights into select attributes of a physical, biological, economic, or social system. A formal representation of the behaviors of system processes, often in mathematical or statistical terms. The basis can also be physical or conceptual (Environment Protection Agency, 2009)

2. WHAT IS ENVIRONMENTAL MODELLING ?

Environmental modelling involves the application of multidisciplinary knowledge to explain, explore and predict the Earth’s response to environmental change, both natural and human-induced. (DBW,2018)

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3. ROLE OF ENVIRONMENT MODELLING

- Improved understanding of environmental systems.
- Developing scientific understanding - through quantitative expression of current knowledge of a system (as well as displaying what we know, this may also show up what we do not know);
- Test the effect of changes in a system;
- Aid decision making, including (i) tactical decisions by managers; (ii) strategic decisions by planners.

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4. TYPE OF MODELS

- Physical modeling
- Empirical models
- Mathematical models

4. TYPE OF MODELS

❖ Physical modeling

- Physical modeling is a way of modeling and simulating systems that consist of real physical components. A physical model is a smaller or larger physical copy of an object.
- Spatial analysis and similarity theories are used in this process to ensure that the model results can be extrapolated to the real system with high accuracy.
- Physical modeling is the main approach of scientists in developing basic theories of the natural sciences.

Water dam model

4. TYPE OF MODELS

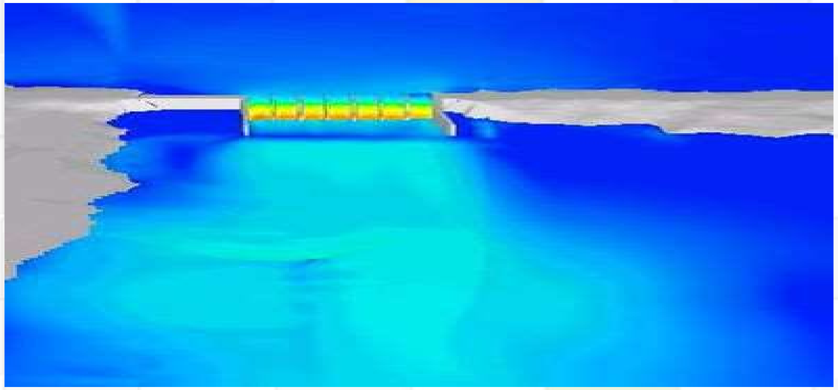
❖ Empirical models

Empirical models describe observed behaviour between variables on the basis of observations alone and say nothing of process. They are usually the simplest mathematical function, which adequately fits the observed relationship between variables. No physical laws or assumptions about the relationships between variables are required. Empirical models have high predictive power but low explanatory depth, they are thus rather specific to the conditions under which data were collected and cannot be generalized easily for application to other conditions

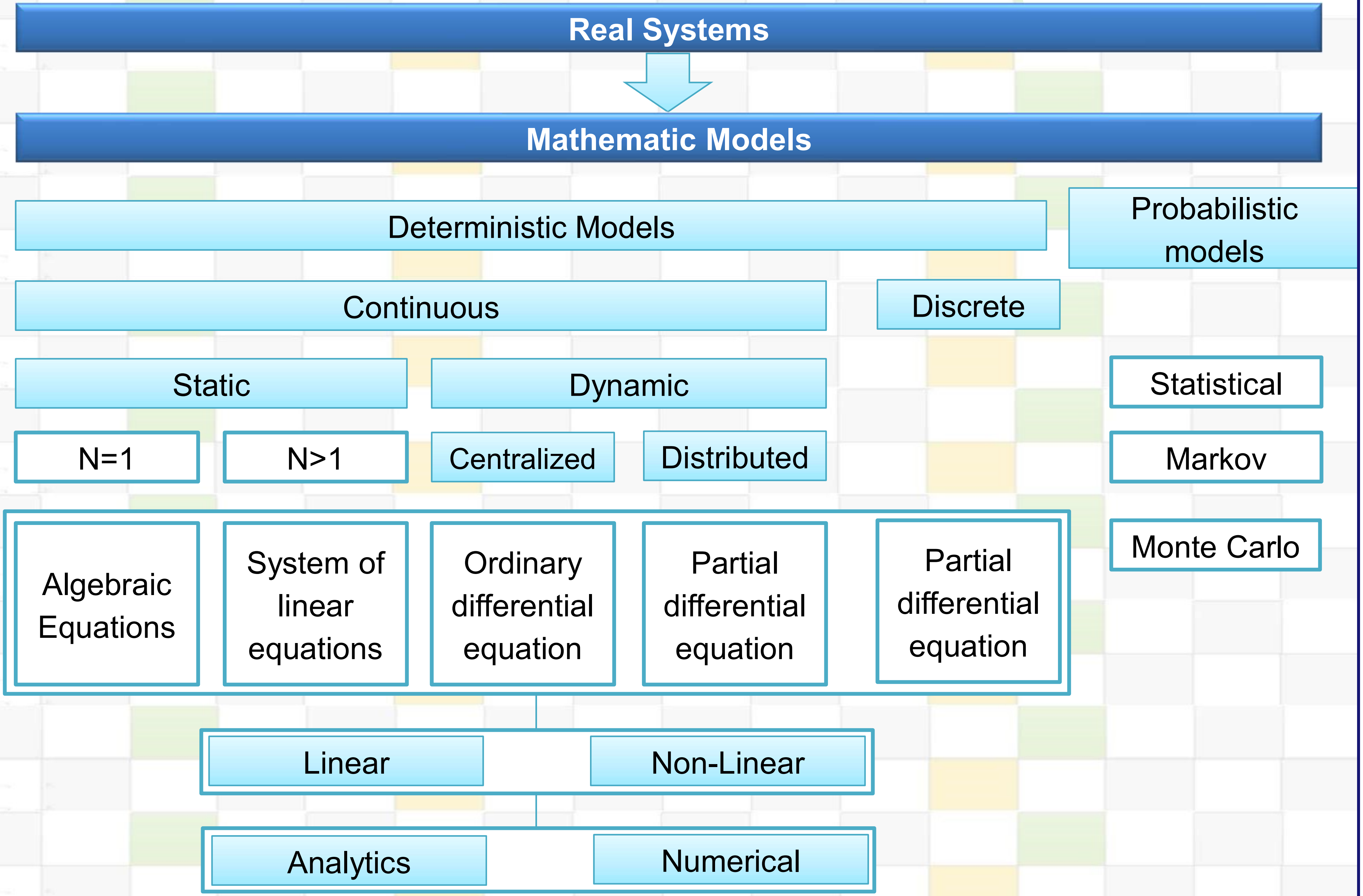
4. TYPE OF MODELS

❖ Mathematical models

- Mathematical model is a representation of real world problem in mathematical form with some simplified assumptions which helps to understand in fundamental and quantitative way
- Mathematical models are much more common and represent states and rates of change according to formally expressed mathematical rules. Mathematical models can range from simple equations to complex software codes applying many equations and rules over time and space discretization. One can further define mathematical models into different types but most models are actually mixtures of many types or are transitional between types.



Dam model is described by a mathematical model



Real Systems

Mathematic Models

Deterministic Models

Probabilistic models

Continuous

Discrete

Static

Dynamic

Statistical

N=1

N>1

Centralized

Distributed

Markov

Algebraic Equations

System of linear equations

Ordinary differential equation

Partial differential equation

Partial differential equation

Monte Carlo

Linear

Non-Linear

Analytics

Numerical

Classification models

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WHAT IS MATHEMATICAL MODELLING?

Definition & Meaning

❖ System and boundary

- A system is a set of one or more related objects, which can be a physical entity with specific properties or characteristics. The system is isolated from its surroundings by boundaries, which can be physical or virtual

❖ Open and Closed, flow/non-flow systems

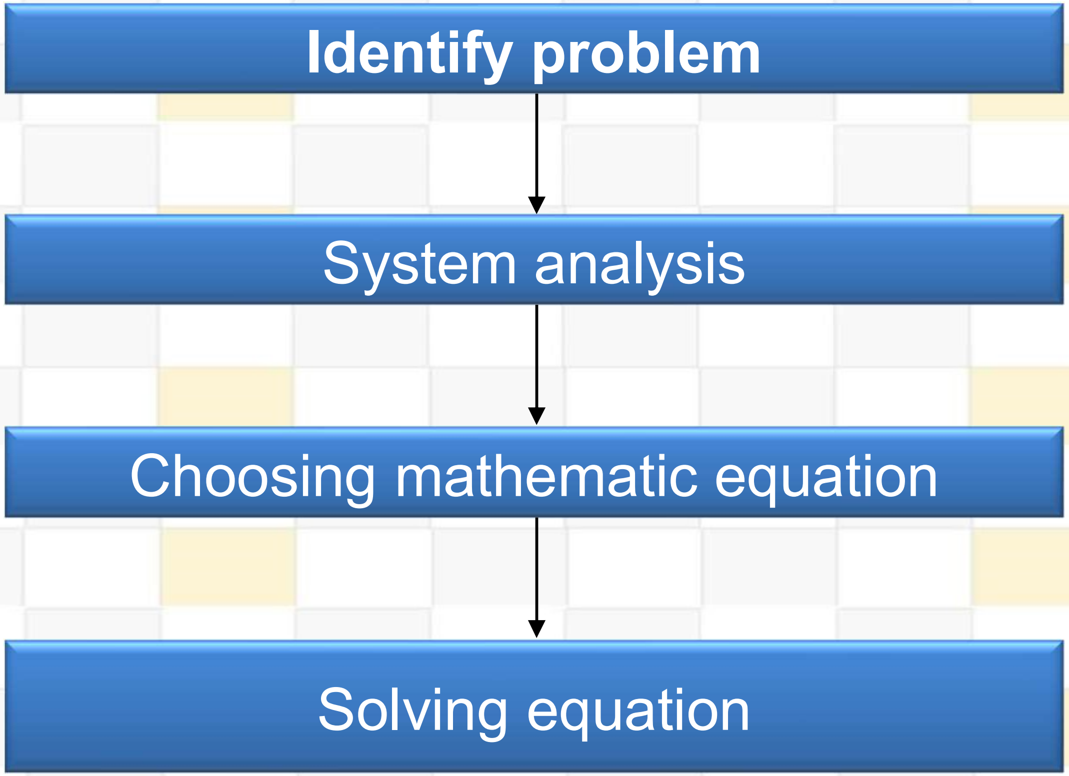
- A closed system is a system that is completely isolated from its environment.
- An open system is a system that has flows of information, energy, and/or matter between the system and its environment, and which adapts to the exchange.
- When the flow of matter does not cross the boundary (but energy can), the system is called a nonflow system. If the material flow can cross the boundary, the system is called a flow system.

Definition & Meaning

❖ Variable, parameter

- A variable is a value that changes freely in time and space (a compartment or flow) and a state variable is one which represents a state (compartment). A constant is an entity that does not vary with the system under study, for example, acceleration due to gravity is a constant in most Earth-based environmental models (but not in geophysics models looking at gravitational anomalies, for example).
- A parameter is a value which is constant in the case concerned but may vary from case to case where a case can represent a different model run or different grid cells or objects within the same model.

APPROACHES TO MATHEMATICAL MODEL BUILDING



Identify problem

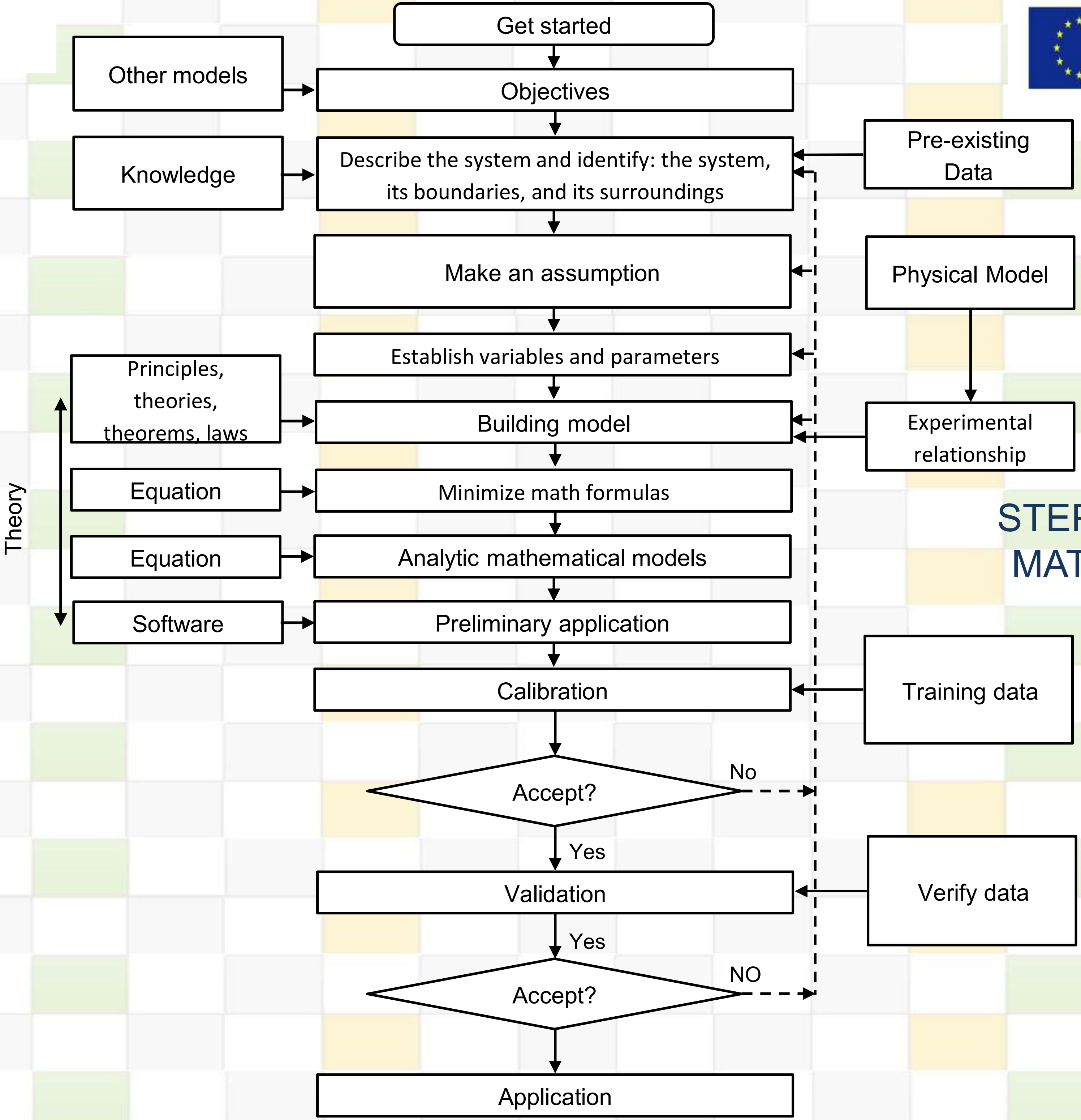
System analysis

Choosing mathematic equation

Solving equation

Step to building mathematical models

STEP TO BUILDING MATHEMMATICAL MODELS



Get started

Other models

Objectives

Knowledge

Describe the system and identify: the system, its boundaries, and its surroundings

Pre-existing Data

Make an assumption

Physical Model

Establish variables and parameters

Principles, theories, theorems, laws

Building model

Experimental relationship

Theory

Equation

Minimize math formulas

Equation

Analytic mathematical models

Software

Preliminary application

Calibration

Training data

Accept?

No

Yes

Validation

Verify data

Yes

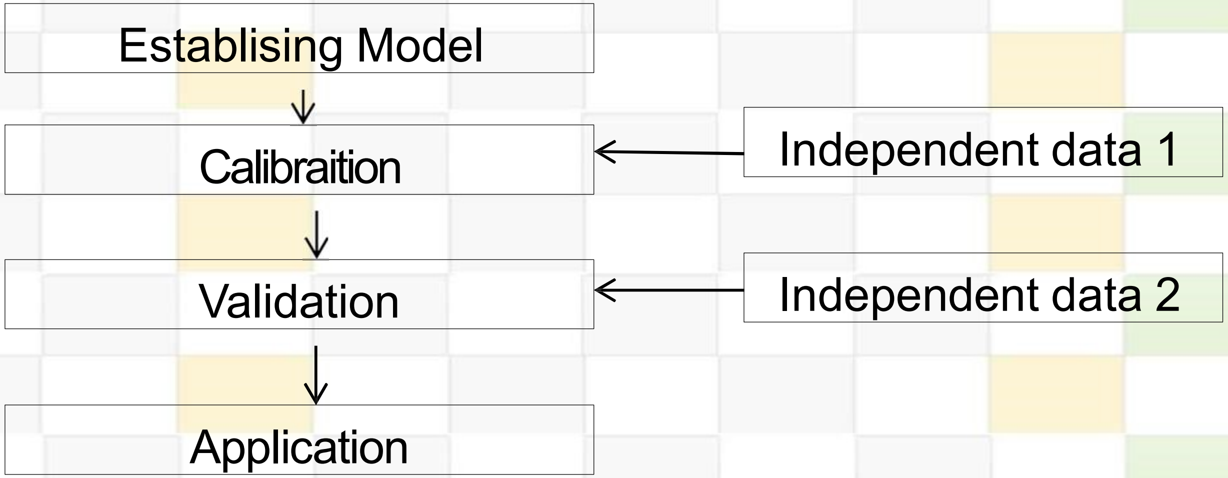
Accept?

NO

Application

APPROACHES TO MATHEMATICAL MODEL BUILDING

CALIBRATION AND VALIDATION


Establising Model

Calibraition

Independent data 1

Validation

Independent data 2

Application

- Calibration is **the iterative process of comparing the model with real system**, revising the model if necessary, comparing again, until a model is accepted (validated)
- Validation is a process of comparing the model and its behavior to the real system and its behavior
- Sensitivity analysis is the process of defining how changes in model input parameters affect the magnitude of changes in model output..

APPROACHES TO MATHEMATICAL MODEL BUILDING

■ Evaluation of simulation results

There are two methods to evaluate model performance

- Graphical method (qualitative)
- Statistical methods (quantitative)

➢ **The calibration model does not represent accurately possibly due to the factors multiply as below:**

- The model is used incorrectly or the model setting is incorrect
- The model is not suitable for this application
- Lack of data to describe the real world
- Measurement data is not reliable

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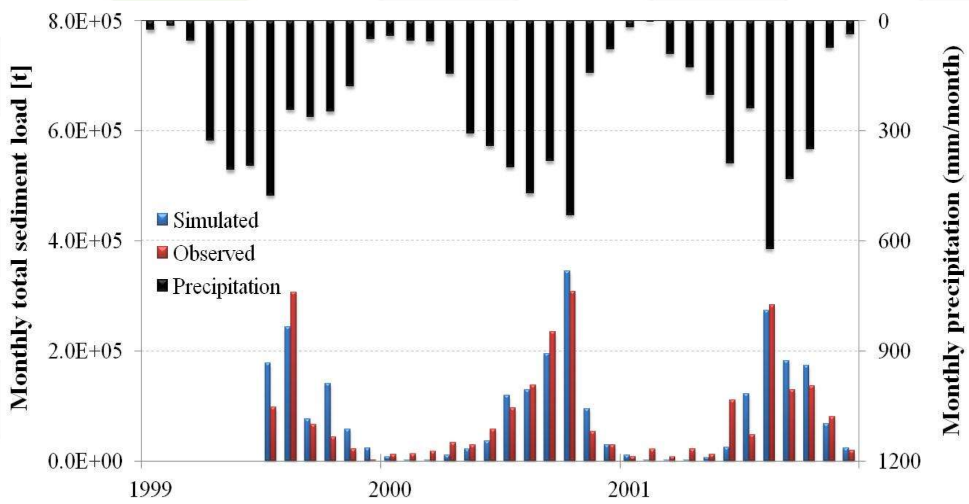
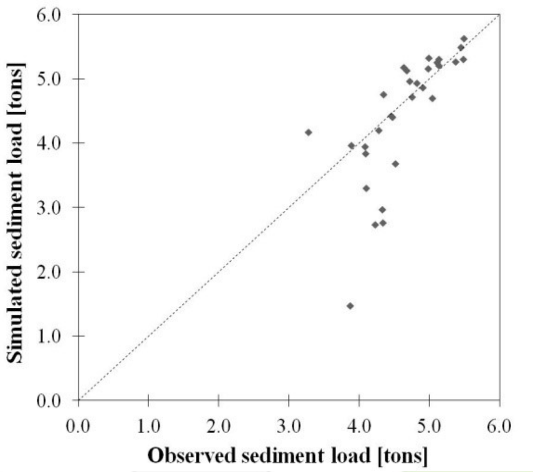
MATHEMATICAL MODEL BUILDING

Evaluation of simulation results

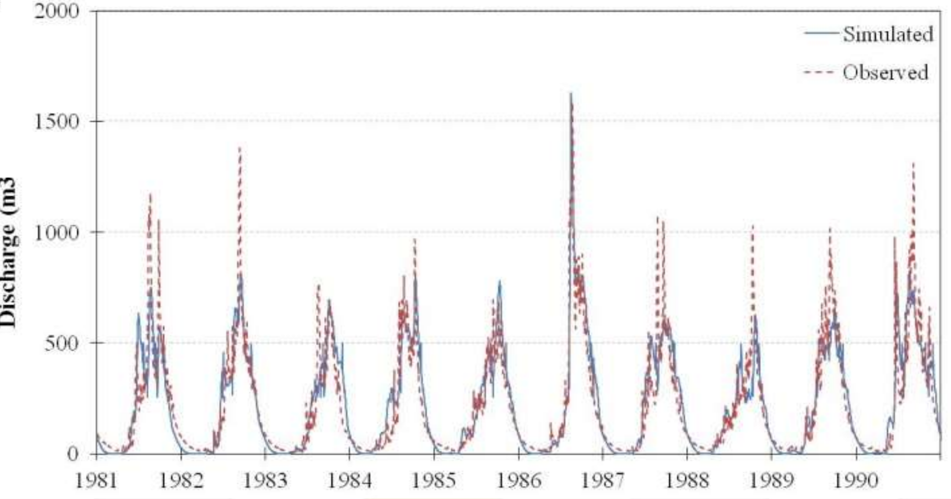
- **Graphical Method**



A. Bar graph



B. Scatter graph



C. Line graph

APPROACHES TO MATHEMATICAL MODEL BUILDING

■ Evaluation of simulation results (cont.)

- **Statistical method**

Percent bias – PBIAS : measures the average tendency of the simulated values to be larger or smaller than their observed ones.

PBIAS → 0: indicating accurate model simulation.

$$PBIAS = \left[\frac{\sum_{i=1}^{n}\left(Y_i^{obs} - Y_i^{sim}\right) \times 100}{\sum_{i=1}^{n}\left(Y_i^{obs}\right)}\right]$$

Y_i^{obs} : observed value

Y_i^{sim}: Simulate value

n: total observed value / Simulate

APPROACHES TO MATHEMATICAL MODEL BUILDING

Evaluation of simulation results (cont.)

- **Statistical method**

Correlation coefficient formulas R^2 are used to find how strong a relationship is between observed and simulate data.

$$R^2 = \frac{\sum_{i=1}^{n}\left(Y_i^{obs} - \bar{Y}^{obs}\right)\left(Y_i^{sim} - \bar{Y}^{sim}\right)}{\sqrt{\sum_{i=1}^{n}(Y_i^{obs} - \bar{Y}^{obs})^2} \times \sqrt{\sum_{i=1}^{n}(Y_i^{sim} - \bar{Y}^{sim})^2}}$$

$\bar{Y}^{obs}$: average value of the series of observed data

$\bar{Y}^{sim}$: average value of the series of simulated data

APPROACHES TO MATHEMATICAL MODEL BUILDING

- **Statistical method**

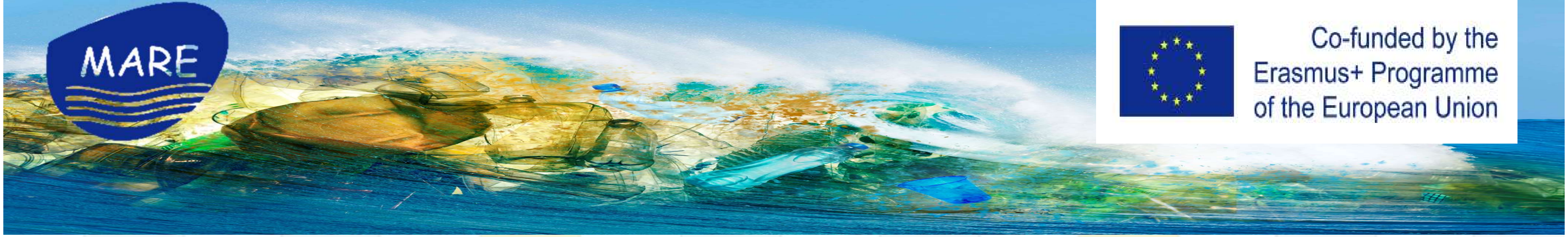
Nash – Sutcliffe (NSE): is a normalized statistic that determines the relative magnitude of the residual variance compared to the measured data variance (Nash and Sutcliffe, 1970). Nash-Sutcliffe efficiency indicates how well the plot of observed versus simulated data fits the 1:1 line. NSE = 1, corresponds to a perfect match of the model to the observed data. NSE = 0, indicates that the model predictions are as accurate as the mean of the observed data, Inf < NSE < 0, indicates that the observed mean is a better predictor than the model.

$$NSE = 1 - \left[\frac{\sum_{i=1}^{n}(Y_i^{obs} - Y_i^{sim})^2}{\sum_{i=1}^{n}(Y_i^{obs} - \bar{Y}^{obs})^2}\right]$$

Properties	**NSE, R^2**	**PBIAS**	
		Flow	***Water - quality***
Very Good	0.75 → 1.00	< ± 10 %	<± 25 %
Good	0.65→ 0.75	± 10 % → ± 15 %	± 25 % → ± 40 %
	0.50→0.65	± 15 % → ± 25 %	± 40 % → ± 70 %
Unsatisfactory	<0.50	> ± 25 %	> ± 70 %

(Moriasi et al., 2007)

THE END



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LECTURE 2:

INTRODUCTION TO MARINE ENVIRONMENT MODELING

Lecturer: Prof. Nguyen Ky Phung
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MSc. Tran Thi Kim

CONTENTS

- ❖ **Purpose of modelling the marine environmental**
- ❖ **Fundamental quantities**
  - Mass and Concentration
  - Rates
- ❖ **Mathematical models**
  - Model Implementations
  - Conservations of Mass and the Mass Balance

PURPOSES OF MODELLING THE MARINE ENVIRONMENTAL

❖ Distribution Load Discharge

- Control the marine environment to achieve a specified environmental quality
- Fishery management and offshores works

❖ Determining Total Maximum Daily Load (TMDL)

- TMDL – Total Maximum Daily Load: the calculation of the maximum amount of a pollutant allowed to enter a waterbody so that the waterbody will meet and continue to meet water quality standards for that particular pollutant. A TMDL determines a pollutant reduction target and allocates load reductions necessary to the source(s) of the pollutant.

PURPOSES OF MODELLING THE MARINE ENVIRONMENTAL

❖ Modelling

- Understanding the processes of the transmission of substances to the marine and ocean environment

❖ Understanding of ecosystem

Understanding of natural system and their reactions to changing conditions.

FUNDAMENTAL QUANTITIES

❖Mass and concentration

- **Mass(m):** The amount of pollutant in a system
- **Concentration(C): conventinally express in metric units.**

$$C = \frac{m}{V}$$

C: Concentration $[ML^{-3}]$

m: mass [M]

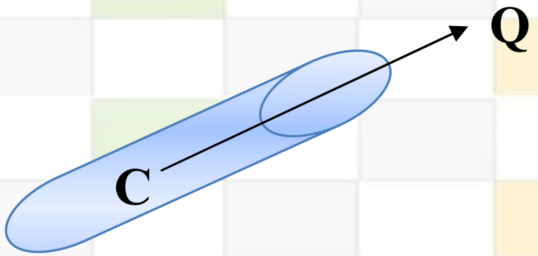
V: Volume $[L^{3}]$

Some water – quality variables along with typical units

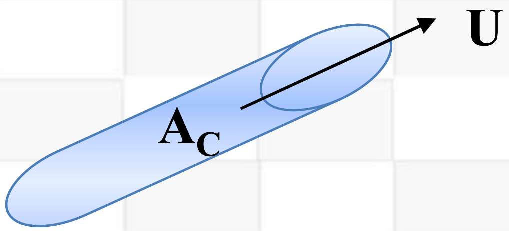
Varibales	Units
Total Disolved solids, salinity	$gL^{-1} \Leftrightarrow kg\ m^{-3} \Leftrightarrow ppt$
DO, BOD, NO_2	$mgL^{-1} \Leftrightarrow g\ m^{-3} \Leftrightarrow ppm$
PO^4, Chlorophyll a, Toxics	$\mu gL^{-1} \Leftrightarrow mg\ m^{-3} \Leftrightarrow ppb$
Toxics	$ngL^{-1} \Leftrightarrow \mu g\ m^{-3} \Leftrightarrow pptr$

FUNDAMENTAL QUANTITIES (cont.)

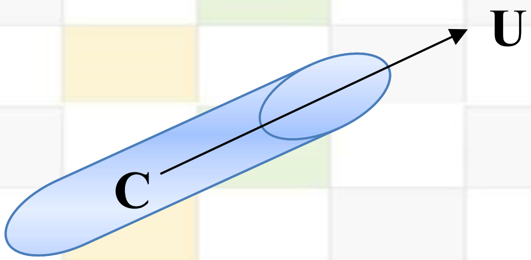
❖Rates



Loading: W = Q.C



Flow: $Q = U.A_c$



Flux: J = U.C

Flux is used to designate the rate of movement of an extensive quantity like mas or heat through a conduit.

$$J = \frac{m}{tA_c} = \frac{W}{A_c}$$

W: Loading [M/T].

C: Concentration[M/L^3].

Q: Flow [L^3/T].

U: Velocity [L/T].

t: time [T].

A_C: Cross-section area[L^2].

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FUNDAMENTAL QUANTITIES (cont.)

Example 1:

A pond having constant volume and no outlet has a surface area A , of 104 m² and a mean depth H of 2 m . It initially has a concentration of 0.8 ppm . Two days later a measurement indicates that the concentration has risen to 1.5 ppm

(a) What was the mass loading rate during this time ?

(b) If you hypothesize that the only possible source of this pollutant was from the atmosphere , estimate the flux that occurred .

FUNDAMENTAL QUANTITIES (cont.)

Example 1

Solution:

(a) The volume of the system can be calculated as:

$$V = A_s H = 10^4 \text{ m}^2 (2\text{m}) = 2 \times 10^4 \text{ m}^3$$

The mass of pollutant at the initial time (t= 0) can be computed as:

$$m = Vc = 2 \text{ x } 10^4 \text{ m}^3 (0.8 \text{ g m}^{-3}) = 1.6 \times 10^4 \text{ g}$$

and at t = 2d is 3.0 x 10^4 g . Therefore the increase in mass is 1.4 x 10^4 g and the mass loading rate is:

$$W=\frac{1.4 \, x \, 10^4 \text{g}}{2\text{d}}= 0.7 \text{ x } 10^4 \text{ g /d}^{-1}$$

(b) The flux of pollutant can be computed as

$$J=\frac{0.7 \, x \, 10^4 \text{g/d}^{-1}}{1 \, x \, 10^4 \text{ m}^2}= 0.7 \text{ g } (\text{m}^2 \text{ d})^{-1}$$

MATHEMATICAL MODELS

Mathematical modes can be represented generally:

$$C = f\,(W, \text{physics, chemistry, biology})$$

→ The cause – effect relationship between loading and concentration depends on the physicals, chemical and biological characteristics of the receiving water.

$$C = \frac{1}{a} W$$

where a = an assimilation factor ($L^3 T^{-1}$) that represents the physics , chemistry , and biology of the receiving water . This Equation is called " linear " because c and W are directly proportional to each other . Consequently if W is doubled , c is doubled . Similarly if W is halved , c is halve

MATHEMATICAL MODELS

❖ Model Implementation

(a) Simulation mode: the model is used to simulate system response (concentration) as a function of a stimulus (loading) and system characteristics (the assimilation factor)

$$C = \frac{1}{a}W$$

(b) Design mode I (Assimilative capacity)

The model can be rearranged to yield : **W = ac**

This implementation is referred to as a " design " mode because it provides information that can be directly used for engineering design of the system. It is formally referred to as an " assimilative capacity " computation because it provides an estimate of the loading required to meet a desirable concentration level or standard . Thus it forms the basis for wastewater treatment plant design . It should also clarify why a is called an " assimilation factor

(c) Design Mode II (environmental modification)

A second design implementation is **a = W/C**. In this case the environment itself becomes the focus of the remedial effort . This equation is formulated to determine how , for a given loading rate , the environment might be modified to achieve the prescribed standard . This type of application is needed when affordable treatment (that is , reduction in W) is not adequate to meet water - quality standards .

MATHEMATICAL MODELS

Example 2: (ASSIMILATION FACTOR)

Lake Ontario in the early 1970s had a total phosphorus loading of approximately 10,500 mta (metric tonnes per annum , where a metric tonne equals 1000 kg) and an in - lake concentration of 21 μg/L. In 1973 the state of New York and the province of Ontario ordered a reduction of detergent phosphate content . This action reduced loadings to 8000 mta .

(a) Compute the assimilation factor for Lake Ontario?

(b) What in - lake concentration would result from the detergent phosphate reduction action ?

(c) If the water - quality objective is to bring in - lake levels down to 10 μg L^{-1}, how much additional load reduction is needed ?

MATHEMATICAL MODELS

Example 2: (ASSIMILATION FACTOR)

Solution:

(a) The assimilation factor can be calculated as

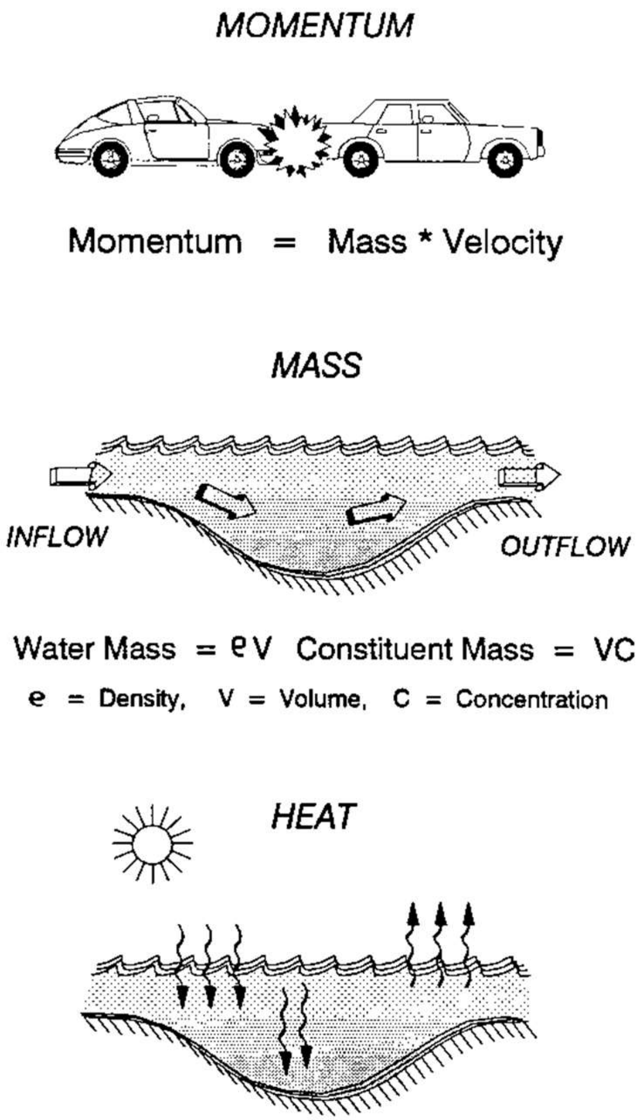
$$a = \frac{W}{c} = \frac{10{,}500\ \text{mta}}{21\mu\text{g}/\text{L}} = 500\,\frac{\text{mta}}{\mu\text{g L}^{-1}}$$

(b) In - lake levels from the phosphorus reduction can be calculated as:

$$c = \frac{W}{a} = \frac{8000\ \text{mta}}{500\frac{\text{mta}}{\mu\text{g L}^{-1}}} = 16\ \mu\text{g L}^{-1}$$

(c) $W = ac = 500\,\frac{\text{mta}}{\mu\text{g L}^{-1}}$ x 10 μg L^{-1} = 5000 mta

CONSERVATIONS OF MASS AND THE MASS BALANCE



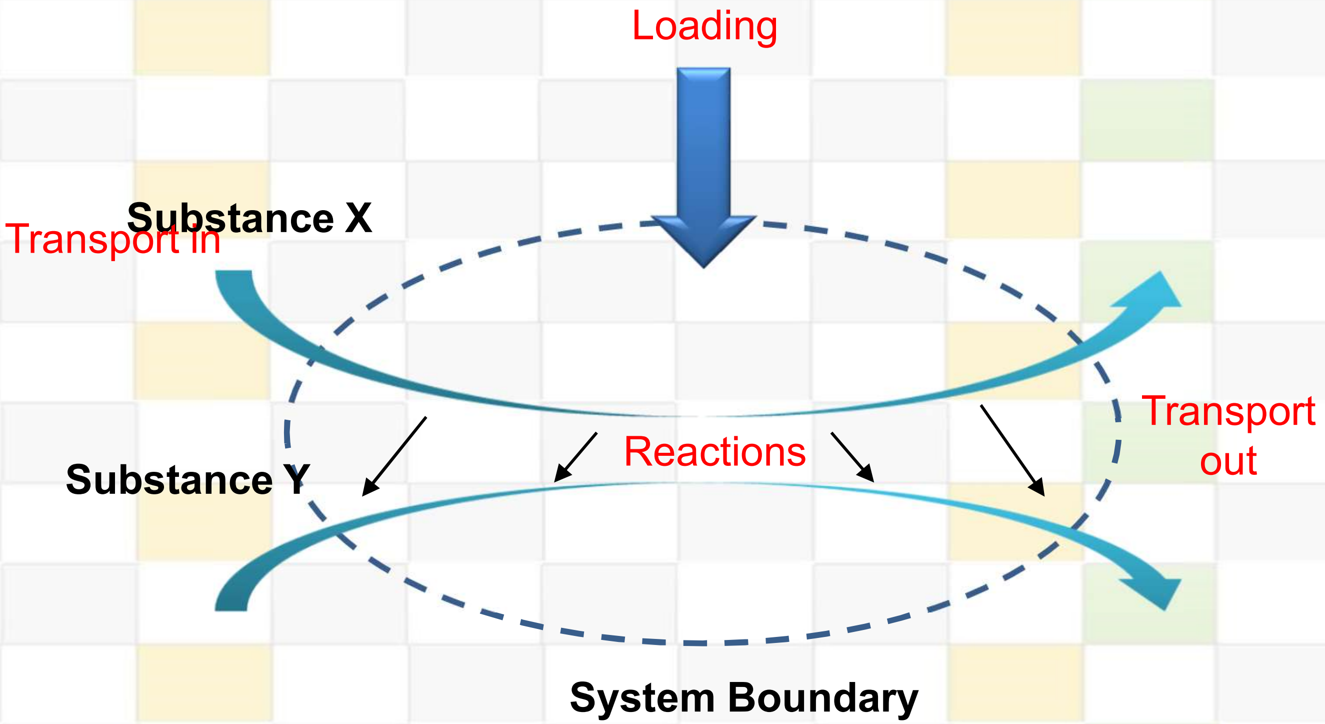
❖ Law of Conservation of Mass

The mass in an isolated system can neither be created nor be destroyed but can be transformed from one form to another (Antoine Lavoisier's 1789)

- Some of Law of Conservation:
  - Mass
  - Momentum
  - Heat

In quantitative terms the principle is expressed as a mass - balance equation that accounts for all transfers of matter across the system's boundaries and all transformations occurring within the system

CONSERVATIONS OF MASS AND THE MASS BALANCE



For a finite period of time this can be expressed as:

Accumulation = loadings transport ± reactions

CONSERVATIONS OF MASS AND THE MASS BALANCE

SYNTHESIS REACTIONS	DECOMPOSITION REACTIONS	NON - REACTIVE SUBSTANCES
• Product of chemical reaction • The growth of algae in the water • Gas absorption • Chemical desorption	• Reactants • BOD • Radioactive decay • Sedimentation • Decomposition of organic matter • Chemical adsorption • Degasification	• Chloride Cl^-, Bromide Br^- • Non-biodegradable organic matter • Metals • Stable isotopes

HISTORICAL DEVELOPMENT OF WATER – QUANLITY MODELS

- 1925-1960: Streeter-Phelps
- 1960-1970: Computerization
- 1970-1977: Biology
- 1977 – present: Toxics

◆ **1925-1960 (streeter-Phelps)**

Problems: untreated and primary effluent

Pollutants: BOD/DO

Systems: streams/estuaries (1D)

Kinetics: linear, feed-forward

Solutions: analytical

◆ **1960-1970** (computerization)

Problems: primary and secondary effluent

Pollutants: BOD/DO

Systems: estuaries/streams(1D/2D)

Kinetics: linear, feed-forward

Solutions: analytical and numerical

◆ **1970-1977 (biology)**

Problems: eutrophication

Pollutants: nutrients

Systems: lakes/estuaries/streams (1D/2D/3D)

Kinetics: nonlinear, feedback

Solutions: numerical

◆ **1977 - nay (toxics)**

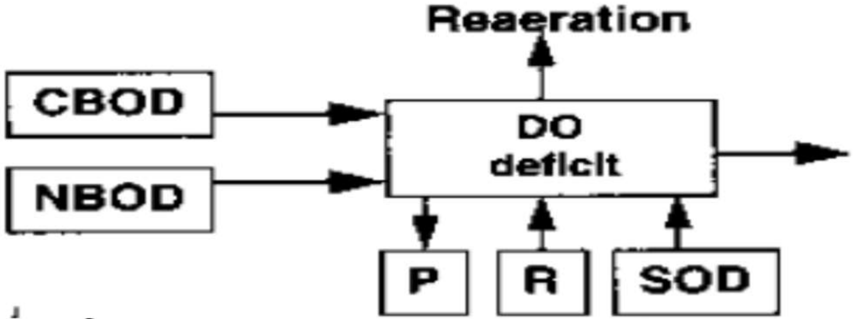
Problems: Toxics

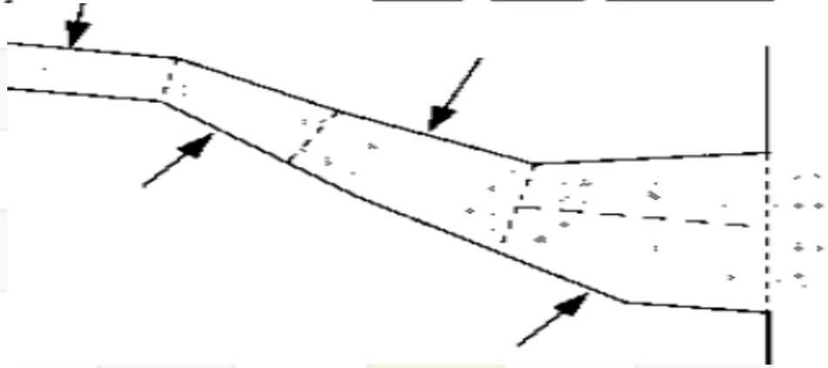
Pollutants: organic,m metals

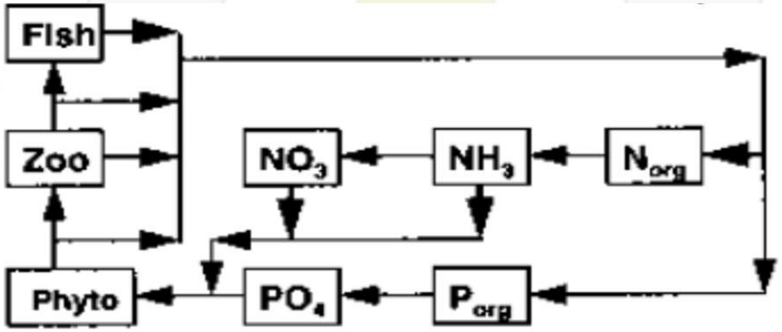
Systems: Sediment-water interactions/food chain ineteraction (lakes/estuaries/stream)

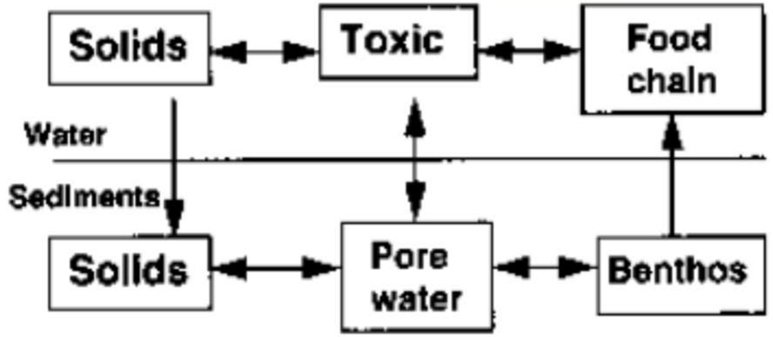
Kinetics: linear, equilibrum

Solutions: numerical and analytical









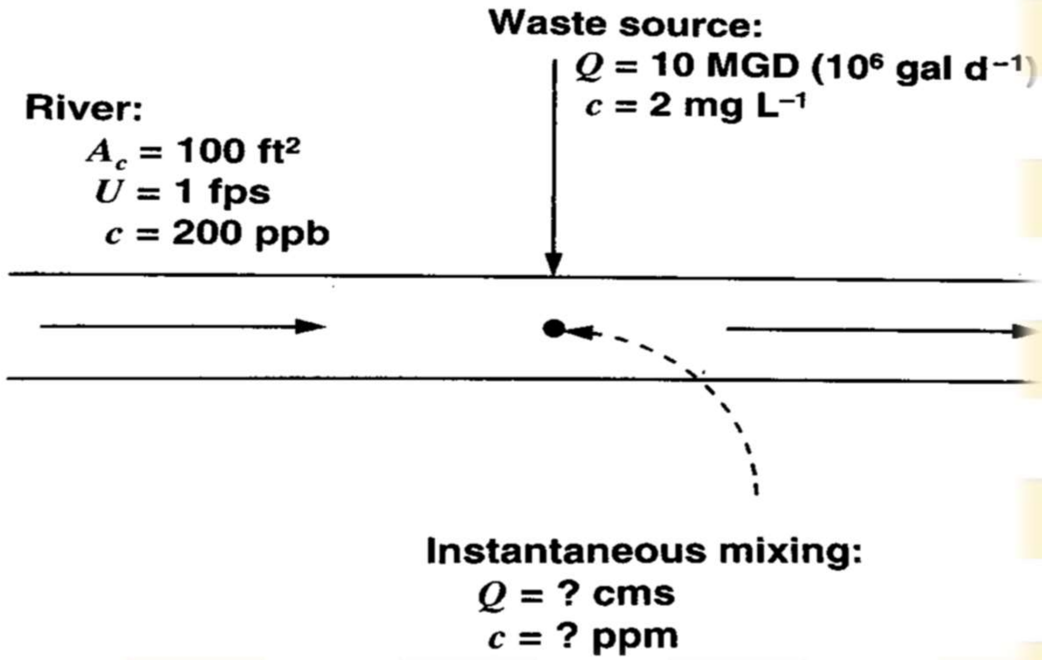
EXERCISES

Ex.2.1: A waste source enters a river as depicted in figure below.

(a) What is the resulting flow rate in m^3s^{-1} (cms)?

(b) If instantaneous mixing occurs, what is the resulting concentration in ppm?

where: 1 gallon=3.785 l, 1ft=0.3048 m, 1ft/s=0.3048 m/s.



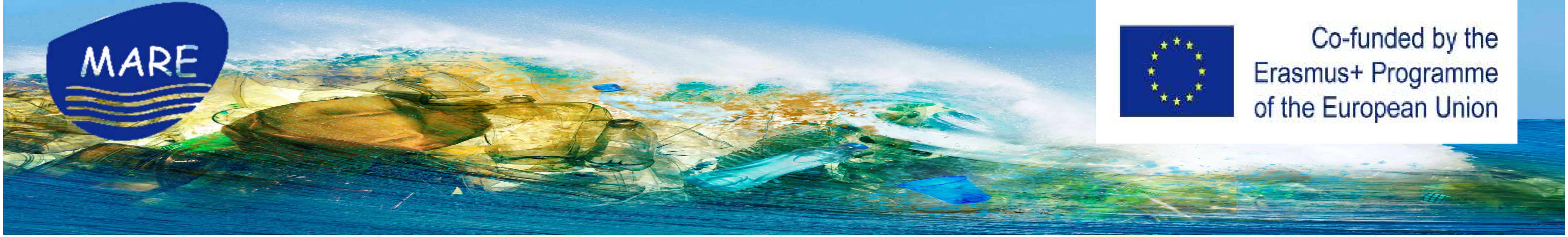
EXERCISES

Ex. 2.2: I the early 1970s Lake Michigan had a total phosphourus loading 6950 mta (mta: metric tones per annum. Metric tone=1000kg;T.e mta=10^3 kg/year) and an in-lake concentration of 8 µg/L

a) Determine the lake’s assimilation factor (km^3/year)

b) What loading rate would be required to bring in-lake levels down to approximately 5 µg/L

--- THE END ---



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LECTURE 3:

DYNAMICS CURRENTS AND TIDES

Lecturer: Prof. Nguyen Ky Phung
MSc. DangThi Thanh Le
MSc. Tran Thi Kim

CONTENTS

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REACTION TYPES

- **Homogeneous reaction:** involve a single phase, (that is, liquid, gas, or solid)
- **Heterogeneous reaction:** involve more than one phase, with reaction usually occurring at the surface between phase
- **Irreversible reaction:** These proceed in a single direction and continue until the reactants are exhausted.
- **Reversible reaction:** can proceed in either direction, depending on the relative concentration of the reactants and the products.

REACTION KINETICS

The law of mass action:

$$\alpha A + \beta B \rightarrow \text{product}$$

Rate of reaction: $$\frac{dC_A}{dt} = -kC_A^{\alpha} C_B^{\beta}$$

Where:

k	constant rate (temperature-dependent)
α	Order with respect to reactant A
β	Order with respect to reactant B
$n = \alpha + \beta$	reaction order

A single reactant:

$$\frac{dC}{dt} = -kC^{n} \quad (*)$$

C = the concentration of the single reactant

n = The order

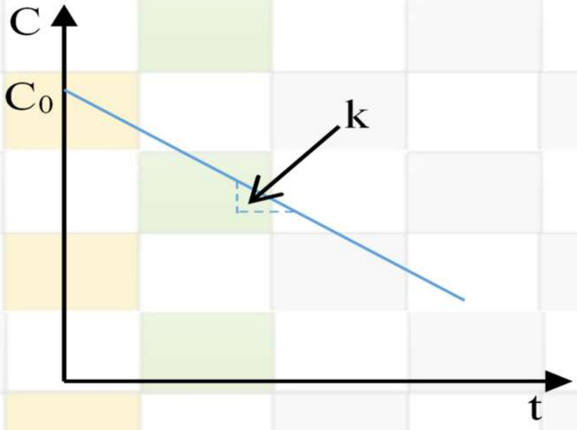
ZERO-, FIRST AND SECOND – ORDER REACTIONS

❖ **Zero - order**(n = 0)

Equation (*): $$\frac{dC}{dt} = -k$$ Where: k has units of $ML^{-3}T^{-1}$

If C = C_0 at t =0, then this equation can be integrated by separation of varibales to yield:

$$C = C_0 - kt$$



- A constant rate of depletion per unit time
- Plot of concentration versus time yields a straight line

ZERO-, FIRST AND SECOND – ORDER REACTIONS

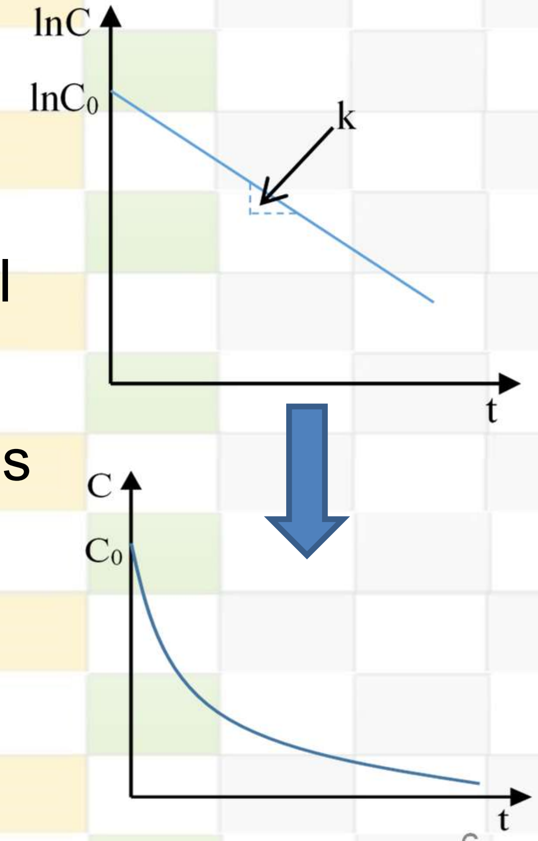
❖ **First - order**(n = 1)

Equation(*): $\frac{dC}{dt} = -kC$ where k has units of T^{-1}

If $C = C_0$ at t =0, then this equation can be integrated by separation of variables to yield: $\ln c - \ln c_o = -kt$

Taking the exponential of both sides gives: $C = C_0 e^{-kt}$

- The concentrations decrease following exponential function
- The concentration curve asymptotically approaches zero with time



Taking te inverse logarithm base - e to base -10

$C = C_0 10^{-k't}$ where $k' = \frac{k}{2.3025}$

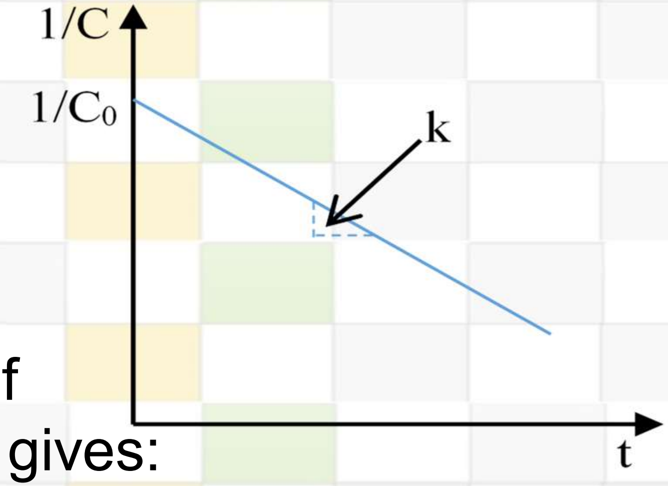
ZERO-, FIRST AND SECOND – ORDER REACTIONS

❖ **Second - order**(n = 2)

Equation(*): $$\frac{dC}{dt} = -kC^2$$ where k has units of $L^3M^{-1}T^{-1}$

If C = C_0 at t =0, then this equation can be integrated by separation of variables to yield

$$\frac{1}{C} = \frac{1}{C_0} + kt$$



- Plot of 1/c versus t should yield a straight line

The above equation can also expressed in term of concentration as a function of time by inverting to gives:

$$C = C_0 \frac{1}{1 + kC_0 t}$$

ZERO-, FIRST AND SECOND – ORDER REACTIONS

- **n order** (n≠1)

Equation(*): $$\frac{dC}{dt} = -kC^n$$

If C = C_0 at t =0, then this equation can be integrated by separation of variables to yield

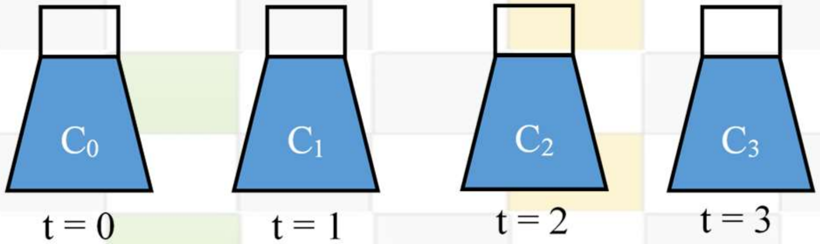
$$\frac{1}{C^{n\text{-}1}} = \frac{1}{C_0^{n-1}} + (n-1)kt$$

- Plot of $1/C^{n\text{-}1}$ versus t should yield a straight line

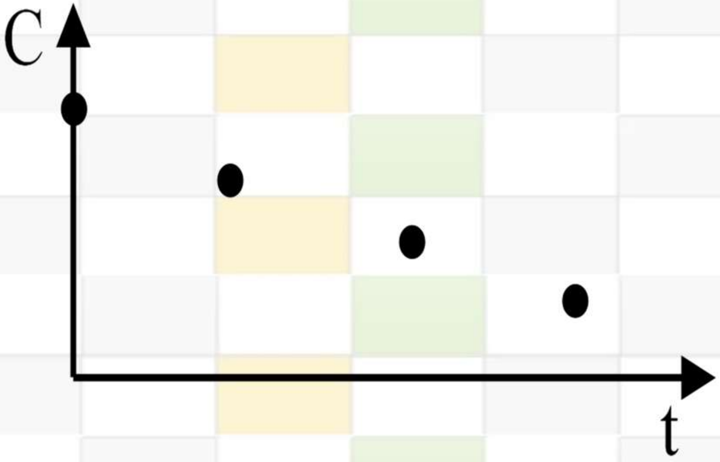
Solution for c: $$C = C_0 \frac{1}{\left[1+(n-1)kC_0^{n-1}t\right]^{1/(n-1)}}$$

ANALYSIS OF RATE DATA

- A simple approach is to measuring concentrations in each bottle over time to develop a relationship between concentrations and time.



t	0	1	2	3
c	C_0	C_1	C_2	C_3



ANALYSIS OF RATE DATA

❖The integral method:

- Step 1: guessing n
- Step 2: Integrating equation (*) to obtain a function,C(t)
- Step 3: Graphical methods are then employes to determine whether the model fits the data adequately

Order	Rate units	Dependent y	Independent x	Intercept	Slope
n=0	$M(L^3T)^{-1}$	C	t	C_0	-K
n=1	T^{-1}	InC	t	InC_0	-K
n=2	$L^3(MT)^{-1}$	1/C	t	$1/C_0$	K
n #1	$(L^3M^{-1})^{n-1}T^{-1}$	C^{1-n}	t	$C_0{}^{1-n}$	(n-1)K

ANALYSIS OF RATE DATA

Example 1: Employ the integral method to determine whether the following data is zero-, first, second – order

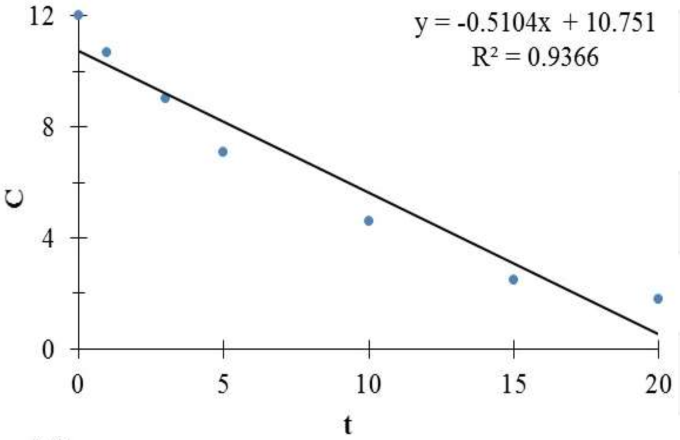
t (day)	0	1	3	5	10	15	20
C (mg/l)	12	10.7	9	7.1	4.6	2.5	1.8

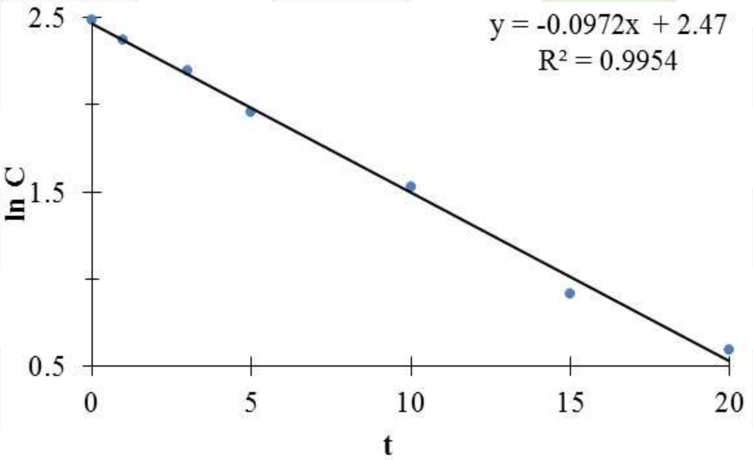
If any of these models seem to hold, evaluate k and c_o

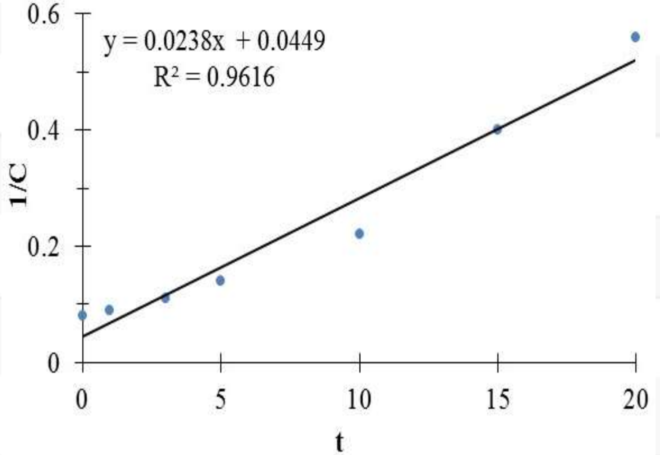
Solution:

To determine whether the following data is zero-, first, second – order, we will evaluate for each order.

t	0	1	3	5	10	15	20
C	12	10.7	9	7.1	4.6	2.5	1.8
ln C	2.48	2.37	2.20	1.96	1.53	0.92	0.59
1/C	0.08	0.09	0.11	0.14	0.22	0.40	0.56







Plots to evaluate whether this reaction is (a) zero-order, (b) first – order, (c) second – order

The best-fit line for this case as below:

$$\ln C = 2.47 - 0.0972t$$ which $R^2 = 0.995$

Therefore the estimates of two model parameter are:

$k = 0.0972$ day^{-1}

$C_0 = e^{2.47} = 11.8$ mg/l

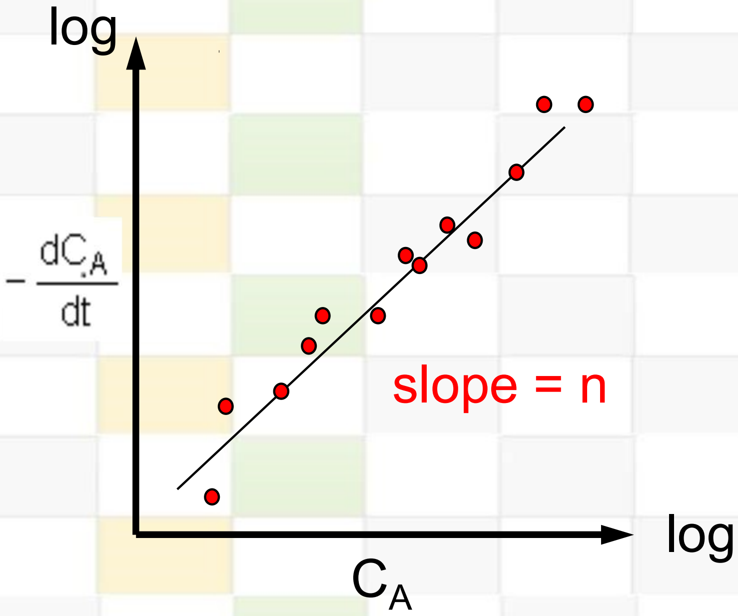
Thus the resulting model is $C = 11.8e^{-0.0972t}$

ANALYSIS OF RATE DATA (cont.)

❖The differential Method

Taking the logarithmic of both sides of Equation(*), to give:

$$\log\left(-\frac{dC}{dt}\right)=\log k+n\log C$$



Plot of log (-dC/dt) versus log C should yield a straight line with a slope of n and an intercept of log k

Numerical differentiation: Finite - Difference approximations to estimate dC/dt.

- Centered Differentiation

$$\frac{dC_i}{dt}\cong\frac{\Delta C}{\Delta t}=\frac{C_{i+1}-C_{i-1}}{t_{i+1}-t_{i-1}}$$

ANALYSIS OF RATE DATA (cont.)

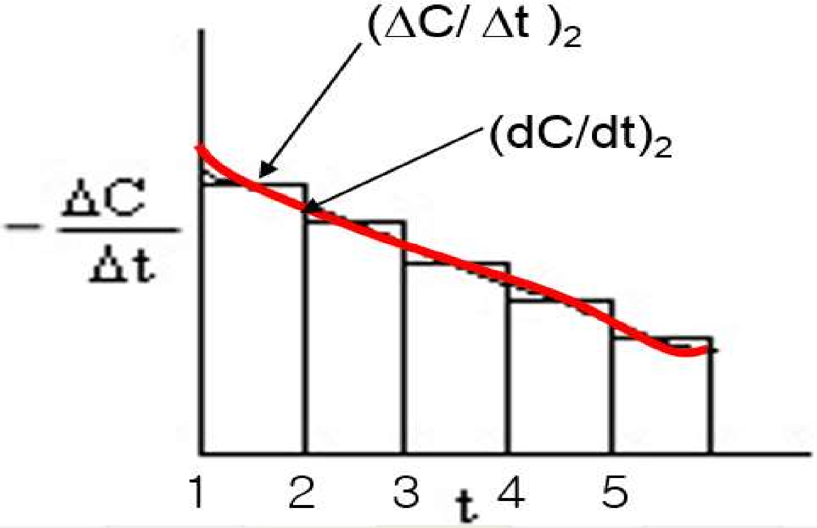
❖ The differential Method (cont.)

t	t_0	t_1	t_2	t_3	t_4	t_5
c	C_0	C_1	C_2	C_3	C_4	C_5

◆ Equal-area differentiation

t	**C**	**Δt**	**ΔC**	**ΔC/Δt**	**dC/dt**
t_1	C_1				$(dC/dt)_1$
		$t_2 - t_1$	$C_2 - C_1$	$(\Delta C/\Delta t)_2$	
t_2	C_2				$(dC/dt)_2$
		$t_3 - t_2$	$C_3 - C_2$	$(\Delta C/\Delta t)_3$	
t_3	C_3				$(dC/dt)_3$
		$t_4 - t_3$	$C_4 - C_3$	$(\Delta C/\Delta t)_4$	
t_4	C_4				$(dC/dt)_4$
		$t_5 - t_4$	$C_5 - C_4$	$(\Delta C/\Delta t)_5$	
t_5	C_5				$(dC/dt)_5$

Drawing smooth curve that best approximates the area under the histogram, try to balance out the histogram areas above and below the drawn curve. Then the derivative estimates at the data points can be read directly from the curve.



ANALYSIS OF RATE DATA (cont.)

❖ The differential Method (cont.)

t	t_0	t_1	t_2	t_3	t_4	t_5
c	C_0	C_1	C_2	C_3	C_4	C_5

◆ Numerical method

First point

$$\left(\frac{dC}{dt}\right)_{t0} = \frac{-3C_0 + 4C_1 - C_2}{2\Delta t}$$

Middle points

$$\left(\frac{dC}{dt}\right)_{t1} = \frac{C_2 - C_0}{2\Delta t}$$

$$\left(\frac{dC}{dt}\right)_{t2} = \frac{C_3 - C_1}{2\Delta t}$$

$$\left(\frac{dC}{dt}\right)_{t3} = \frac{C_4 - C_2}{2\Delta t}$$

$$\left(\frac{dC}{dt}\right)_{t4} = \frac{C_5 - C_3}{2\Delta t}$$

End point

$$\left(\frac{dC}{dt}\right)_{t5} = \frac{C_3 - 4C_4 + 3C_5}{2\Delta t}$$

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ANALYSIS OF RATE DATA (cont.)

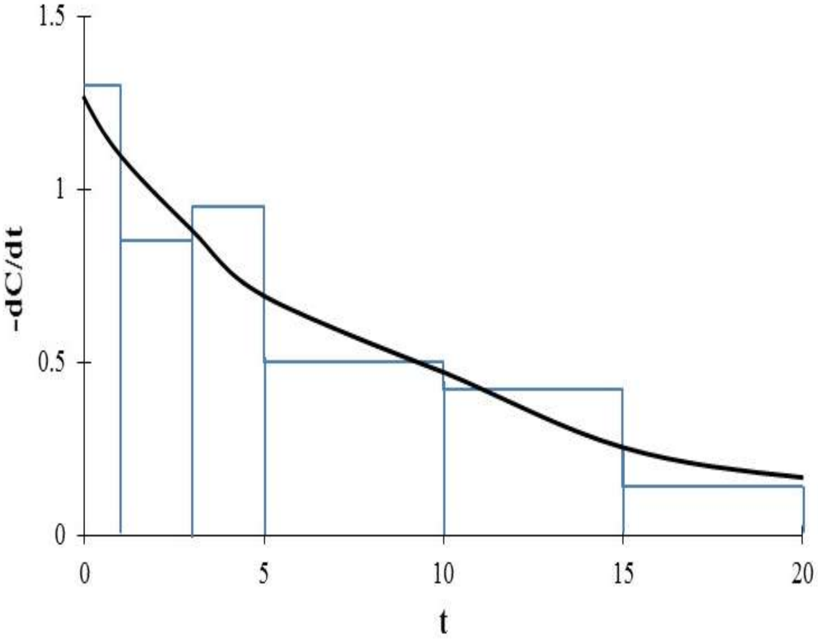
Example 2: Use the differential method to evaluate the order and the constant for the data from Example 1. Use equal-area differentiation to smooth the derivative estimates

t (d)	0	1	3	5	10	15	20
C (mg/l)	12	10.7	9	7.1	4.6	2.5	1.8

Solution:

Determine derivative estimate from time series of concentration

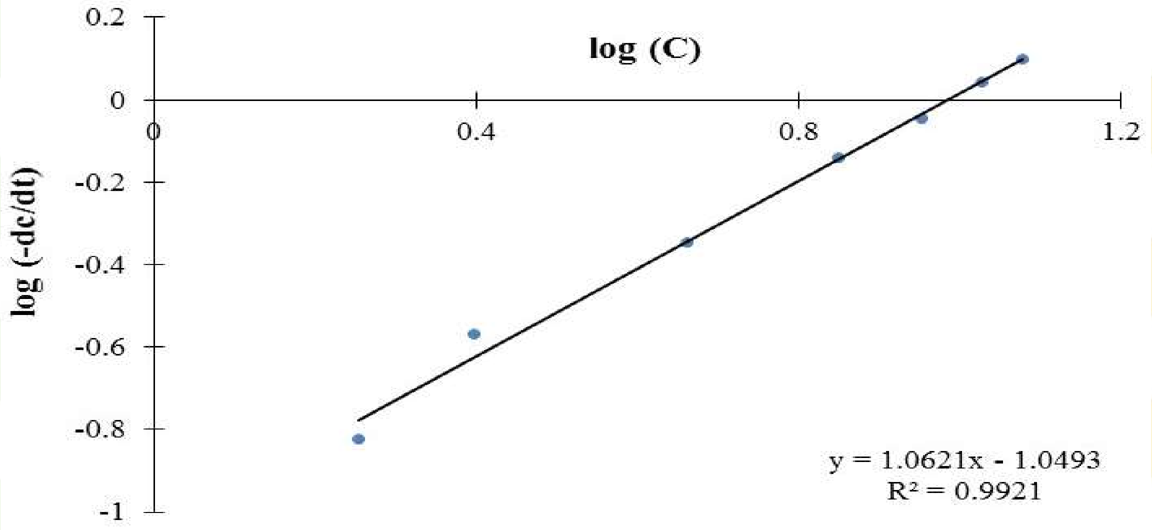
t	C	$-\Delta C/\Delta t$	$-dC/dt$	logC	log(-dC/dt)
(day)	(mg/l)	mg/l/d			
0	12.0		1.25	1.08	0.1
		1.3			
1	10.7		1.1	1.03	0.04
		0.85			
3	9.0		0.9	0.95	-0.05
		0.95			
5	7.1		0.72	0.85	-0.14
		0.50			
10	4.6		0.45	0.66	-0.35
		0.42			
15	2.5		0.27	0.40	-0.57
		0.14			
20	1.8		0.15	0.26	-0.82



Equal-area differentiation

The best-fit line for this case is

$$\log\left(-\frac{dC}{dt}\right) = -1.049 + 1.062\log C \quad \text{với} \quad R^2 = 0.9921$$



Plot log (-dC/dt) versus log (C)

Therefore the estimates of two model parameter are:

- $n = 1.062$ (First - order)
- $k = 10^{-1.049} = 0.089$/day

Thus the differential approach suggest that a first – order model is a valid approximately

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ANALYSIS OF RATE DATA (cont.)

❖ The method of Initial Rates

- There are cases where reactions occur in which complications arise over time. For example a significant reverse reaction might occur. Further some reactions are very slow and the time required for the complete experiment might be prohibitive.
- Using data from the beginning stages of the experiment to determine the rate constant and order
- The differential method

Taking the logarithm of the negative of Eq. (*):

$$\log\left(-\frac{dC_0}{dt}\right) = \log k + n\log C_0$$

Plot log(-dC0/dt) versus log (C_o) should yield a straight line can be used to estimate k and n, the slope provides an estimate of the order, whereas the intercept provides an estimate logarithm

ANALYSIS OF RATE DATA (cont.)

❖The method of Half -lives

The half-lives of a reaction is the time it takes for the concentration to drop to one-half of its initial value. In other words.

$$C(t_{50}) = 0.5C_0$$

If C = C_0 at t =0, then this equation (*) can be integrated by separation of variables to yield

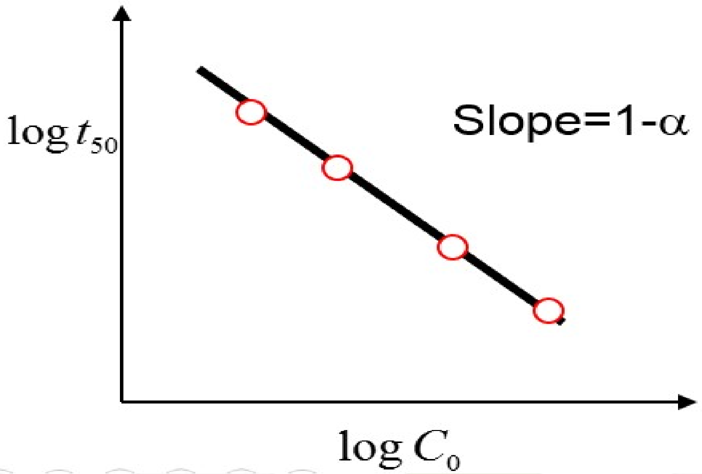
$$t = \frac{1}{kC_0^{n-1}(n-1)}\left[\left(\frac{C_0}{C}\right)^{n-1} - 1\right]$$

Combining two equations above to give:

$$t_{50} = \frac{2^{n\text{-}1} - 1}{k(n-1)}\frac{1}{C_0^{n-1}}$$

Taking the logarithm of this equation provides a linear relationship

$$\log t_{50} = \log\frac{2^{n\text{-}1} - 1}{k(n-1)} + (1\text{ - }n)\log C_0$$



ANALYSIS OF RATE DATA (cont.))

❖The method of Half -lives (cont.)

For general case with response time t_ϕ, where ϕ is percent reduction:

$$t_\varphi = \frac{\left[100/\left(100-\varphi\right)\right]^{n-1}-1}{k(n-1)} \frac{1}{C_0^{n-1}}$$

TEMPERATURE EFFECTS

The rates of most reaction in natural waters increase with temperature. A more rigorous quantification of the temperature dependence is provided by the Arhenius equation.

$$k(T_a) = Ae^{\frac{-E}{RT_a}}$$

A	a preexponential or frequency factor
E	Activation energy (J.mole^{-1})
R	the gas constant(8.314 J.mole^{-1}.K^{-1})
T_a	absolute temperature (K).

Compare the reaction rate constant at two different temperatures

$$\frac{k(T_{a2})}{k(T_{a1})} = e^{\frac{E(T_{a2}-T_{a1})}{RT_{a2}T_{a1}}}$$

$T_{a1}.T_{a2}$ = const

$$\theta = e^{\frac{E}{RT_{a2}T_{a1}}}$$

$$\frac{k(T_{a2})}{k(T_{a1})} = \theta^{T_{a2}-T_{a1}}$$

Compare the reaction rate constant at 20ºC: $k(T) = k(20)\theta^{T\text{-}20}$

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TEMPERATURE EFFECTS

Example 3: Evaluation of temperature dependency of reaction. A laboratory provides you with the flowing results for a reaction

T_1 = 4ºC k_1 = 0.12 ngày^{-1}

T_2 = 16ºC k_2 = 0.20 ngày^{-1}

(a) Evaluate θ for this reaction

(b) Determine the rate at 20ºC

Solution:

(a) Taking the logarithm and raise the results to power of 10 to give

$$\theta = 10^{\frac{\log k(T_2) - \log k(T_1)}{T_2 - T_1}}$$

Substituting the data gives:

$$\theta = 10^{\frac{\log 0.12 - \log 0.20}{4 - 16}} = 1.0435$$

(b) The rate at 20ºC

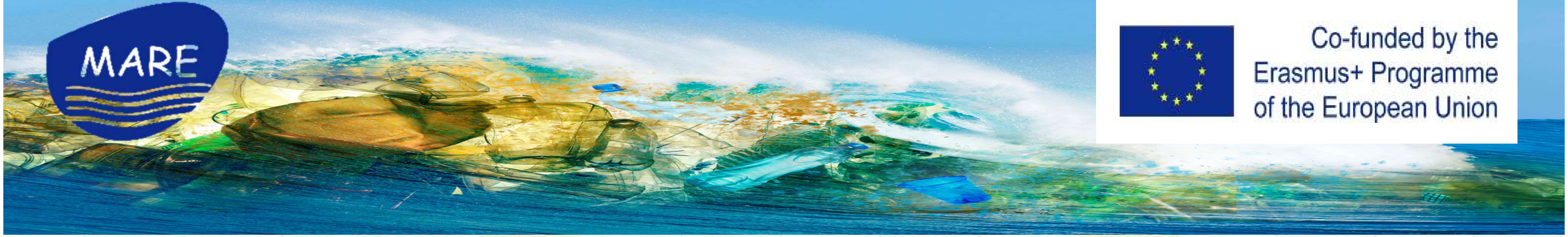
$$k(20) = 0.20 \times 1.0435^{20-16} = 0.237\ (\text{ngày})$$

BÀI TẬP

1. We design an experiment and determine the oxygen concentration as follows:

t(d)	0	2	5	10	20	30	40	50	60	70
C(mg/l)	10	8.4	6.5	4.4	2.3	1.6	1.3	1.2	1.1	1.1

Determine the order and the rate of the reaction?



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LECTURE 4:

CURRENTS DYNAMICS

Lecturer: Prof. Nguyen Ky Phung
MSc. DangThi Thanh Le
MSc. Tran Thi Kim

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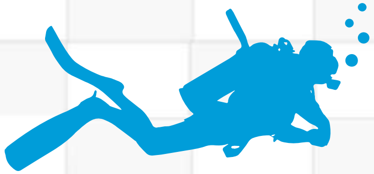

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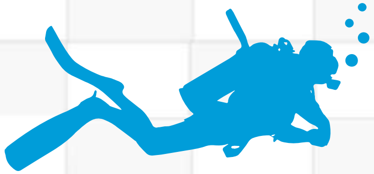
GENERAL CONCEPT AND CURRENTS CLASSIFICATION



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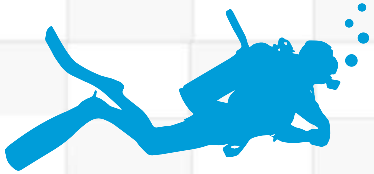
General concept

- *Current is the* horizontal movement of water in a well- defined pattern. Currents are generally measured in meters per second or in knots (1 knot = 1.85 kilometers per hour or 1.15 miles per hour). Currents are driven by three main factors: The rise and fall of the tides, Wind, Thermohaline circulation.



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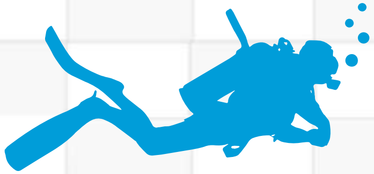
The role of currents

❖ Currents play a huge role in marine ecosystem:

- Increases water exchange.
- Redistribution of temperature, salt level.
- Transforming the shore.
- Moving sea ice.
- At the same time, it strongly affects the atmospheric circulation and climate of parts of the Earth.



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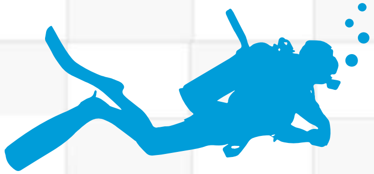
Currents classification

❖Currents can be classified according to the following basic characteristics:

- According to the factors or forces that driven the currents
- According to stability.
- According to the distribution depth.
- According to the nature of movement.
- According to the physiological nature of the water mass.

❖ In currents theories, classifying currents by factors or forces that cause currents is considered the main classification.



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Currents classification

- ❖ According to the forces that cause the currents, the currents can be divided into three main groups
- **Tidal currents**: Tides create a current in the oceans, which are strongest near the shore, and in bays and estuaries along the coast. Tidal currents change in a very regular pattern and can be predicted for future dates. In some locations, strong tidal currents can travel at speeds of eight knots or more.
- **Wind currents**: Winds drive currents that are at or near the ocean's surface. Near coastal areas winds tend to drive currents on a localized scale and can result in phenomena like coastal upwelling. On a more global scale, in the open ocean, winds drive currents that circulate water for thousands of miles throughout the ocean basins.
- **Gradient current** is the current caused by the horizontal gradient of hydrostatic pressure that occurs when the sea surface is on its side.

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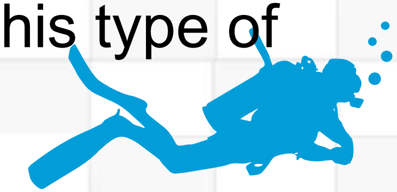
Currents classification

❖ According to the depth of distribution can be divided into:

- Surface currents are also driven by global wind systems fueled by energy from the sun. Factors like wind direction and the Coriolis effect play a role.
- Deep currents, also known as thermohaline circulation, result **from differences in water density**. These currents occur when cold, dense water at the poles sinks
- The current close to the bottom is the flow observed in the water layer close to the bottom. Bottom friction strongly affects this type of flow.



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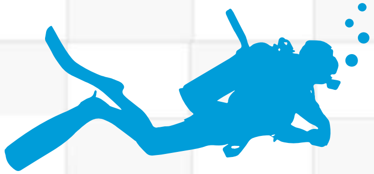
Currents classification

❖ According to the nature of movement one divides the flow into:

- Meandering currents
- Straight currents
- Curved currents

❖ According to the physiological nature of the water mass in the flow people divided into:

- Hot current
- Cold currents
- Salty currents
- Light currents



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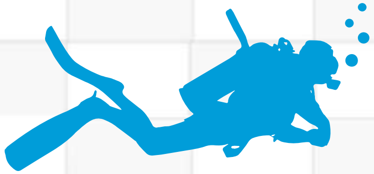
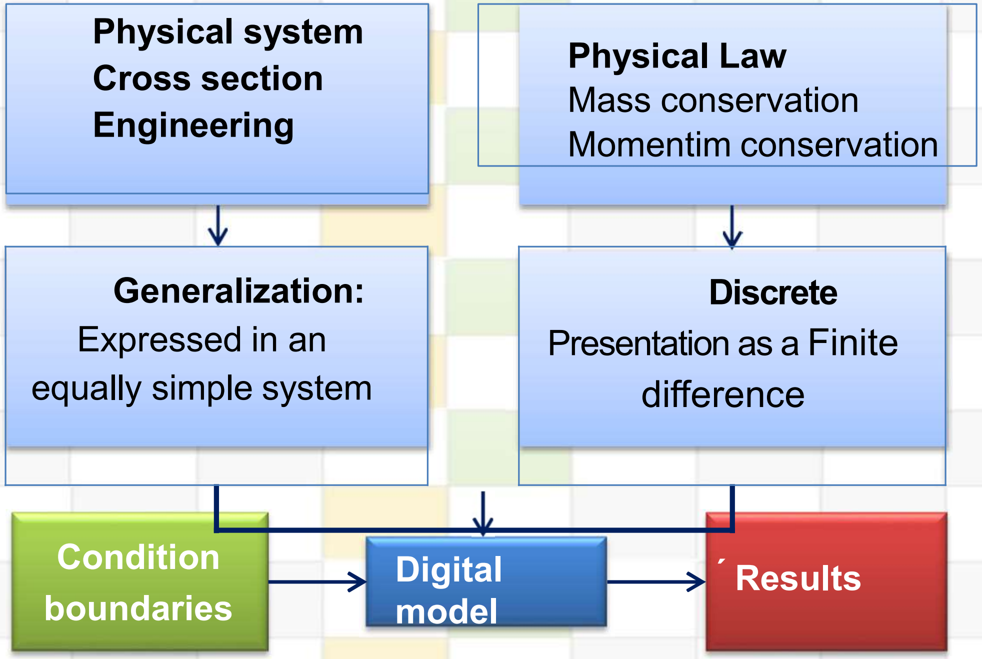
HYDRODYNAMICS



HYDRODYNAMICS

- **Study the movement of the fluid and the forces acting on it.**
- **Flow modeling relies on three basic factors:**
  - Partial differential equations representing the laws of physics
  - Finite difference diagram for generating systems of algebraic equations
  - Algorithms to solve these equations



Physical system
Cross section
Engineering

Physical Law
Mass conservation
Momentim conservation

Generalization:
Expressed in an equally simple system

Discrete
Presentation as a Finite difference

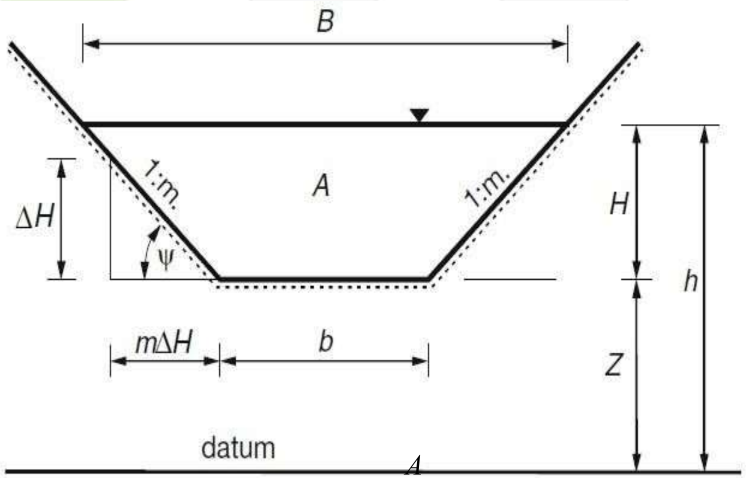
Condition boundaries

Digital model

´ Results

HYDRODYNAMICS (Cont)

Geometric features of the channel cross section



Bottom width : b
- Surface width : B
- Cross section area : A
- Perimeter : P
- Hydraulic radius : R = A/P
- Slope roof coefficient : m=cotgΨ

Flow: $$Q = \iint^{A} u \times dA$$

Mean velocity $$U = \frac{\mathrm{Q}}{\mathrm{A}} = \frac{1}{\mathrm{A}} \iint^{A} u \times dA$$

Flux $$q = \int_{Z}^{h} u \times dz = U \times H$$

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HYDRODYNAMICS (cont)

Meaning in terms of energy.

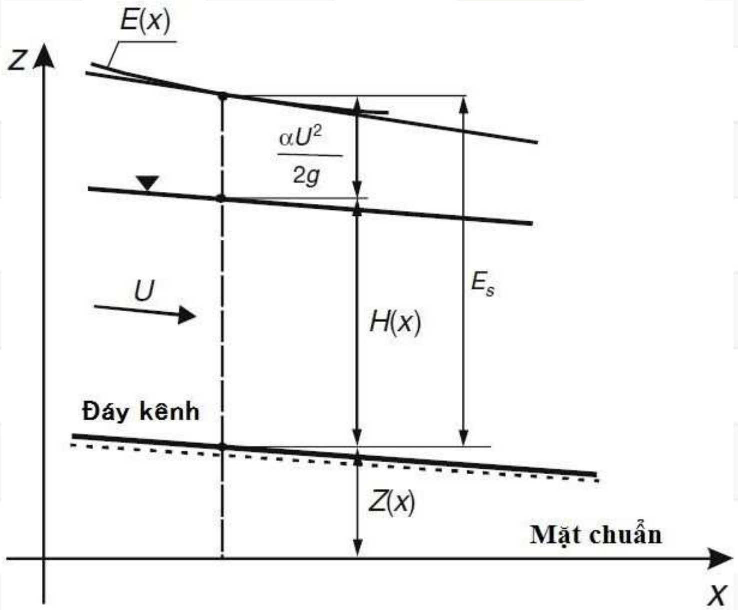
Bernoulli equation (Mechanical energy equation)

$$E_{sl} = \boxed{z_{sl}} + \boxed{\frac{p_{sl}}{\gamma}} + \boxed{\frac{u_{sl}^2}{2g}}$$

Elavation Presure Kinetic

Energy equation for open channel

$$E = a + h + \frac{\alpha U^2}{2g} = H + \frac{\alpha U^2}{2g}$$



α - kinetic energy correction coefficient

$$\alpha = \frac{1}{U^3 A}\iint_A u^3 \times dA$$

HYDRODYNAMICS (Cont)

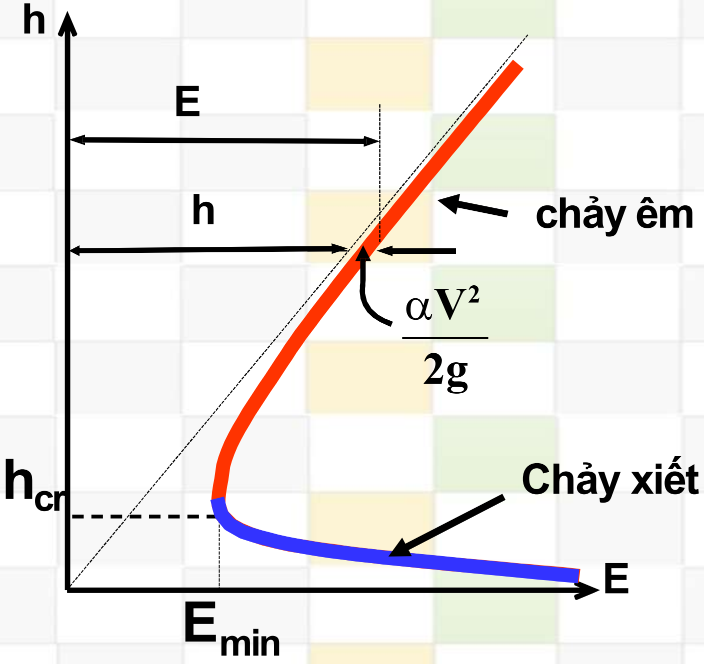
Determine the critical position :

$$\left(\frac{dE}{dh}\right)_{h=h_{cr}} = 0 \Rightarrow \frac{\alpha Q^2}{g} = \frac{A_{cr}^3}{B_{cr}}$$

$$\frac{dE}{dh} = \frac{d}{dh}\left(h + \frac{\alpha U^2}{2g}\right) = 1 - \frac{\alpha U^2}{2gH}$$

$$= 1 - Fr^2$$

Froude: $Fr = \frac{U}{\sqrt{gH}}$ $\frac{\text{force of inertia}}{\text{gravitation}}$



- Smooth flow : Fr < 1
- Fast flow: Fr > 1
- Demarcation flow: Fr = 1

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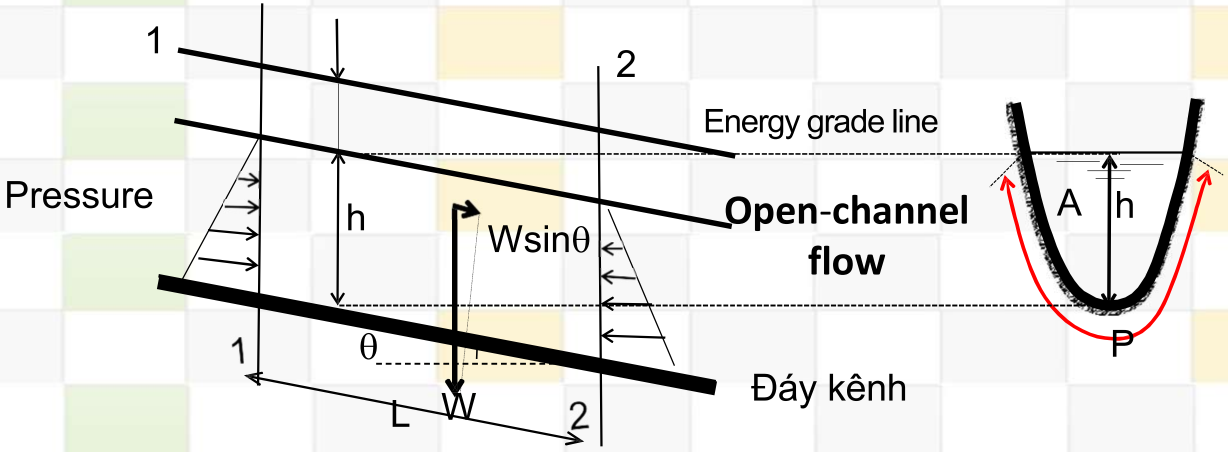
HYDRODYNAMICS (Cont)

Chezy and Manning equations

Chezy equation $U = C_C\left(R\times s\right)^{1/2}$

C_c – Chezy discharge coefficient



Applied force:

- Gravity
- Friction force
- Pressure = 0

Gravity: $W_x = W\sin\theta = \gamma AL\sin\theta = \gamma ALs$

Friction force: $F_x = LPkU^2$

Force balance
:

$$\Rightarrow U = \left(\frac{\gamma}{k}\right)^{1/2}\sqrt{Rs}$$

$$C_C = \sqrt{\frac{\gamma}{k}}$$

- Manning Equation: $U = \frac{1}{n}R^{2/3}s^{1/2}$ với $C_C = \frac{1}{n}R^{1/6}$

HYDRODYNAMICS (Cont)

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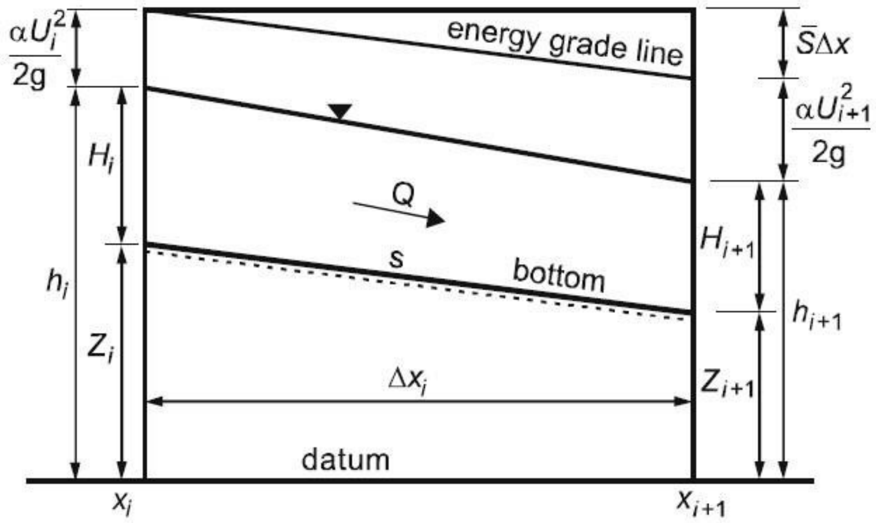
Energy equation for fluid

$$h_i + \frac{\alpha U_i^2}{2g} = h_{i+1} + \frac{\alpha U_{i+1}^2}{2g} + \overline{S}\Delta x_i$$

where:

h_i, h_{i+1}	Mực nước tại vị trí i và i+1
U_i, U_{i+1}	Velocity at i and i+1 positions
$\overline{S}$	Energy slope
Δx_i	The distance between the two locations.



Energy slope (friction slope)):

$$S = \frac{n^2 U^2}{R^{4/3}} = \frac{n^2 Q^2}{R^{4/3} A^2}$$

HYDRODYNAMICS (Cont)

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Saint-Venant equation

Approach

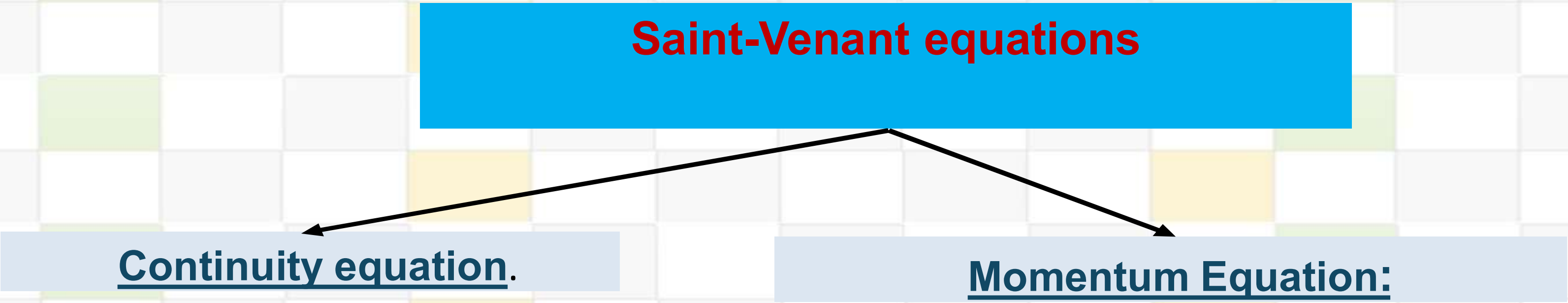
Simplifying actual flow processes in a 1-dimensional problem

➢ **Assumption**

- Flow is one-dimensional
- Hydrostatic pressure prevails and vertical accelerations are negligible
- Streamline curvature is small.
- Bottom slope of the channel is small.
- Manning's equation is used to describe resistance effects
- The fluid is incompressible

HYDRODYNAMICS (Cont)

- **The basic law of physics**



Saint-Venant equations

Continuity equation.

Momentum Equation:

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HYDRODYNAMICS (Cont.)

- **Saint-Venant equations**
- **Continuity equation**

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q$$

Momentum equation

$$\frac{\partial Q}{\partial t} + \frac{\partial(\alpha \frac{Q^2}{A})}{\partial x} + gA\frac{\partial h}{\partial x} + \frac{gQ|Q|}{C^2 AR} = 0$$

Trong đó:

q - input flow per unit of channel length.

q > 0 – inflow to the control volume

q < 0 - outflow to the control volume

--- THE END ---

LECTURE

MODELLING THE MARINE ENVIRONMENT

Lecturer: Prof. Nguyen Ky Phung
MSc. Dang Thi Thanh Le

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Lecture 5

FUNDAMENTAL EQUATIONS DESCRIBING CURRENTS AND BOUNDARY CONDITIONS

Lecturer: Professor. Nguyen Ky Phung
Ms. Dang Thi Thanh Le

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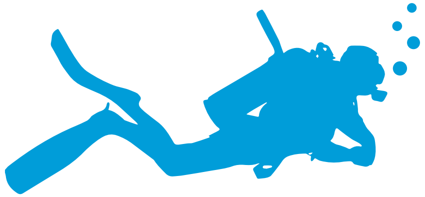


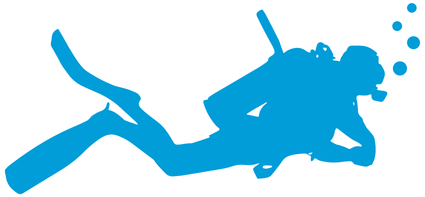
FUNDAMENTAL EQUATIONS DESCRIBING CURRENTS AND BOUNDARY CONDITIONS

The state of the liquid is completely determined, if at each point of the liquid at any time completely identified:

Pressure P(x,y,z,t);

Density ρ(x,y,z,t),

Velocity with components u(x,y,z,t), v(x, y, z,t), w(x,y,z,t).



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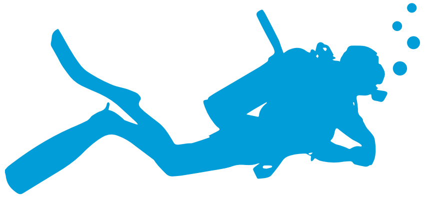
Equations of Motion for Perfect and Viscous Fluids

In general, a **particle** is being **acted upon** by the **following forces**

- ✓ **Inertia forces** (-γ = -dV/dt for a unit of mass);
- ✓ External forces F (for a unit of mass and with components X,Y,Z)
- ✓ Mutual friction force (pressure, viscous): R.



is the time differential of a definite liquid particle, and the liquid particle's speed is a function of both time and sp

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Equations of Motion for Perfect and Viscous Fluids

According to the Dalambe principle of the equilibrium of the dx dy dz water particle under the effect of those three forces, we have:

$$\rho \mathrm{dx} \cdot \mathrm{dy} \cdot \mathrm{dz} \cdot \frac{\mathrm{dV}}{\mathrm{dt}} = \rho \cdot \mathrm{F} \cdot \mathrm{dxdydz} + \mathrm{R} \quad (5.1)$$

Projecting the equation (1.25) onto the coordinate axes and dividing by ρ (for non-compressible liquids), we have:

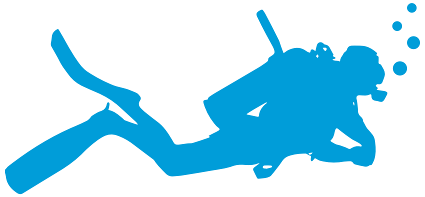
$$\begin{aligned} \frac{du}{dt} &= X - \frac{1}{\rho}\frac{\partial P}{\partial x} + \upsilon \nabla u \\ \frac{dv}{dt} &= Y - \frac{1}{\rho}\frac{\partial P}{\partial y} + \upsilon \nabla v \qquad (5.2) \\ \frac{dw}{dt} &= Z - \frac{1}{\rho}\frac{\partial P}{\partial z} + \upsilon \nabla w \end{aligned}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$; υ = const is a kinetic coefficient, for the perfect liquid: v = 0;

$$\begin{aligned} \frac{\mathrm{du}}{\mathrm{dt}} &= \mathrm{X} - \frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial \mathrm{x}} \\ \frac{\mathrm{d}v}{\mathrm{dt}} &= \mathrm{Y} - \frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial \mathrm{y}} \qquad (5.3) \\ \frac{\mathrm{dw}}{\mathrm{dt}} &= \mathrm{Z} - \frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial \mathrm{z}} \end{aligned}$$

Where: $\frac{\mathrm{d}}{\mathrm{dt}}$ is the time differential of a definite liquid particle, and the liquid particle's velovcity is a function of both time and space, so we have:

$$\frac{\mathrm{du}}{\mathrm{dt}} = \frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \mathrm{w}\frac{\partial \mathrm{u}}{\partial \mathrm{z}}$$



Equations of Motion for Perfect and Viscous Fluids

Therefore the motion equation has form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial P}{\partial x}+\upsilon\nabla u$$

$$\frac{\partial \mathrm{v}}{\partial \mathrm{t}}+\mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{x}}+\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{y}}+\mathrm{w}\frac{\partial \mathrm{v}}{\partial \mathrm{z}}=\mathrm{Y}-\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial \mathrm{y}}+\upsilon\nabla \mathrm{v} \qquad (5.4)$$

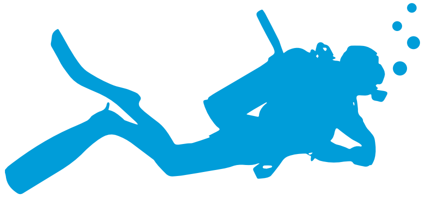
$$\frac{\partial \mathrm{w}}{\partial \mathrm{t}}+\mathrm{u}\frac{\partial \mathrm{w}}{\partial \mathrm{x}}+\mathrm{v}\frac{\partial \mathrm{w}}{\partial \mathrm{y}}+\mathrm{w}\frac{\partial \mathrm{w}}{\partial \mathrm{z}}=\mathrm{Z}-\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial \mathrm{z}}+\upsilon\,\nabla \mathrm{w}$$

Equation (5.4) is the Navie-St. equation for the viscous fluids. For the ideal liquid we have:

$$\upsilon\nabla \mathrm{u}=\upsilon\nabla \mathrm{v}=\upsilon\nabla \mathrm{w}=0$$

The above equations are nonlinear. We're going to linearize them if we ignore space acceleration. Movement is considered stable (or stop) if the velocity at any given time does not depend on the time or local acceleration:

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}}=\frac{\partial \mathrm{v}}{\partial \mathrm{t}}=\frac{\partial \mathrm{w}}{\partial \mathrm{t}}=0$$



CONTINUITY EQUATION

Considering the continuity equation of the fluid, i.e. we consider the continuous properties of that fluid. Assuming there is a fluid volume factor $\delta x\, \delta y\ \delta z$, consider the volume of liquid entering and exiting this volume during the period δt.

In the direction of the Ox axis, the mass of fluid entering that volume: $(\rho u)_x \delta t \delta y \delta z$

and the mass of fluid that comes out of volume is: $(\rho u)_{x+\delta x} \delta t \delta y \delta z$

So, after the period δt $\delta x\, \delta y\ \delta z$ là: Ox axis movement will increase the amount of fluid in the volume factor

$$(\rho u)_x \delta z \delta y \delta t - (\rho u)_{x+\delta x} \cdot \delta z \delta y \delta t = -\frac{\partial(\rho \cdot u)}{\partial x} \delta x \delta y \delta z \delta t$$

Similarly, in the direction of the Oy axis and the Oz axis we also have:

$$-\frac{\partial(\rho . v)}{\partial y} \delta x \delta y \delta z \delta t; -\frac{\partial(\rho \cdot w)}{\partial z} \delta x \delta y \delta z \delta t$$

CONTINUITY EQUATION

According to the law of mass conservation, the total volume of liquids entering and exiting volume δx δy δz must be by changing the volume of fluid during that time.:

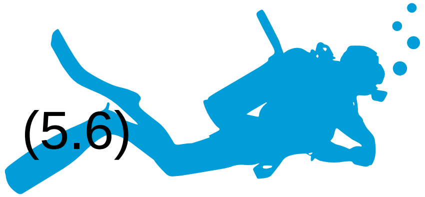
$$\left(\rho+\frac{\partial\rho}{\partial \mathrm{t}}\partial \mathrm{t}\right)\delta \mathrm{x}\delta \mathrm{y}\delta \mathrm{z}-\rho\delta \mathrm{x}\delta \mathrm{y}\delta \mathrm{z}=\frac{\partial\rho}{\partial \mathrm{t}}\delta \mathrm{t}\delta \mathrm{x}\delta \mathrm{y}\delta \mathrm{z}$$

$$=-\frac{\partial(\rho \mathrm{u})}{\partial \mathrm{x}}\delta \mathrm{x}\delta \mathrm{y}\delta \mathrm{z}\mathrm{t}-\frac{\partial(\rho \mathrm{v})}{\partial \mathrm{y}}\delta \mathrm{x}\delta \mathrm{y}\delta \mathrm{z}\delta \mathrm{t}-\frac{\partial(\rho \mathrm{w})}{\partial \mathrm{z}}\delta \mathrm{x}\delta \mathrm{y}\delta \mathrm{z}\delta \mathrm{t}$$

$$\frac{\partial\rho}{\partial \mathrm{t}}+\frac{\partial(\rho \mathrm{u})}{\partial \mathrm{x}}+\frac{\partial(\rho \mathrm{v})}{\partial \mathrm{y}}+\frac{\partial(\rho \mathrm{w})}{\partial \mathrm{z}}=0$$

Or $$\frac{\partial\rho}{\partial \mathrm{t}}+\mathrm{div}(\rho.\mathrm{V})=0 \qquad (5.5)$$

The equation (5.5) is the continuity equation of compressed liquid. In practical calculations it is common to view liquids as uncompressed ρ = const: $\frac{\partial\rho}{\partial \mathrm{t}}=0$



Continuity equation is constantly in form: $$\mathrm{div}V=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (5.6)$$

SALT CONSERVATION EQUATION

Consider the amount of salt that goes beyond the limits of a volume factor δx δy δz follow the Ox axis at a time of ration x during the period δt là $(S.\rho.u)_x$ δyδzδt, The amount of salt passing through the opposite side is $(S.\rho.u)_{x+\delta x}\delta y\delta z\delta t$.

Therefore, the movement in the direction of the Ox axis has the excess salt.: $-\frac{\partial(\rho\cdot u.S)}{\partial x}\delta x\delta yz\delta t$

In the direction of the axes Oy and Oz also: $\frac{\partial(\rho\cdot \mathrm{u.S})}{\partial \mathrm{y}}\delta\mathrm{x}\delta\mathrm{y}\delta\mathrm{z}\delta\mathrm{t}$; $-\frac{\partial(\rho\cdot \mathrm{w}\cdot \mathrm{S})}{\partial \mathrm{z}}\delta\mathrm{x}\delta\mathrm{y}\delta\mathrm{z}\delta\mathrm{t}$

Order the initial amount of salt in volume δx δy δz là ρS.δx δy δz, after δt the amount of salt in the volume factor is: $\left(\rho\cdot \mathrm{S}+\frac{\partial(\rho\cdot \mathrm{S})}{\partial \mathrm{t}}\delta \mathrm{t}\right)\delta\mathrm{x}\delta\mathrm{y}\delta\mathrm{z}$

According to the law of conservation: $+\frac{\partial}{\partial \mathrm{t}}(\rho\cdot \mathrm{u}\cdot \mathrm{S})+\frac{\partial}{\partial \mathrm{y}}(\rho\cdot \mathrm{v}\cdot \mathrm{S})+\frac{\partial}{\partial \mathrm{Z}}(\rho\cdot \mathrm{w}\cdot \mathrm{S})=0$ (5.7)

When using continuity equations (1.29) we have: $\frac{\partial \mathrm{S}}{\partial \mathrm{t}}+\mathrm{u}\frac{\partial \mathrm{S}}{\partial \mathrm{x}}+\mathrm{v}\frac{\partial \mathrm{S}}{\partial \mathrm{y}}+\mathrm{w}\frac{\partial \mathrm{S}}{\partial \mathrm{t}}=0$ (5.8)

When S(x,y,z) = const , so $\frac{\partial \mathrm{S}}{\partial \mathrm{t}}=0$, therefore: $\mathrm{u}\frac{\partial \mathrm{S}}{\partial \mathrm{x}}+\mathrm{v}\frac{\partial \mathrm{S}}{\partial \mathrm{y}}+\mathrm{w}\frac{\partial \mathrm{S}}{\partial \mathrm{z}}=0$ (5.9)

The heat conservation equation is received in the same way.:

$$\boldsymbol{\frac{\partial T}{\partial t}+\mathbf{u}\frac{\partial T}{\partial x}+\mathbf{v}\frac{\partial T}{\partial y}+\mathbf{w}\frac{\partial T}{\partial z}=0} \qquad \textbf{(5.10)}$$

SEAWATER STATE EQUATION

Seawater is a compressed liquid, i.e. its density changes. The dependence of the specific volume α and the density of the water on the state parameters: T temperature, S salinity and P pressure are indicated by the state equation. The general form of the state equation is as follows:

$$\rho = \rho(\mathrm{T,S,P}) \qquad (5.11)$$

$$\alpha = \alpha(\mathrm{T,S,P}) \qquad (5.12)$$

Determines the specific volume change of seawater as a function of state parameters:

$$\mathrm{d}\alpha = \left(\frac{\partial\alpha}{\partial \mathrm{T}}\right)_{\mathrm{SP}} \mathrm{dT} + \left(\frac{\partial\alpha}{\partial \mathrm{S}}\right)_{\mathrm{TP}} \mathrm{dS} + \left(\frac{\partial\alpha}{\partial \mathrm{P}}\right)_{\mathrm{TS}} \mathrm{dP}. \qquad (5.13)$$

If you divide all the components of (1.37) by a particular unit volume α_0 the pre-fractions of temperature, salt and pressure will be:

- Thermal expansion coefficient:

$$\mathrm{K_T} = \frac{1}{\alpha_0}\left(\frac{\partial\alpha}{\partial \mathrm{T}}\right)_{\mathrm{S.P}} \qquad (5.14)$$

- Salt compression coefficient:

$$\mathrm{K_s} = -\frac{1}{\alpha_0}\left(\frac{\partial\alpha}{\partial \mathrm{S}}\right)_{\mathrm{T.P}}$$

- Compression resistance coefficient of density:

$$\mathrm{K_p} = -\frac{1}{\alpha_0}\left(\frac{\partial\alpha}{\partial \mathrm{P}}\right)_{\mathrm{T.S}}$$

SEAWATER STATE EQUATION

Then the expression (5.14) is often called the equation of state in the differential form of seawater

$$\frac{d\alpha}{\alpha_0} = K_T dT - K_s dS - K_P dP \qquad (5.15)$$

Currents theories often use simpler systems. The simplest of these is the Businesq approximation (the linear dependence of density on temperature): $\frac{\rho}{\rho_0} = C_1 + C_2 \frac{T}{T_0}$ (5.16)

and the linear dependence of density on temperature and salt (Linheikin, Robinson and Stommel, Bryan and Kox):

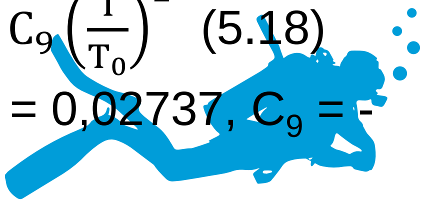
$$\frac{\rho}{\rho_0} = C_3 + C_4 \frac{T}{T_0} + C_5 \frac{S}{S_0}. \qquad (5.17)$$

Where T is temperature, S is salt, ρ_0 is the average density of seawater; T_0 and S_0 are the average values of temperature and salt. When atmospheric pressure is equal 1 at, T_0=17,5^{0}C, S_0 = $35^0/_{00}$,

ρ_0 = 1,02541 g/cm^3 thì các hệ số có giá trị: C_1 = 1,00266; C_2 = C_4 = -0,00317; C_3 = 0,97529; C_5 = 0,02737.

More precise dependence of density on temperature and salt: $\frac{\rho}{\rho_0} = C_6 + C_7 \frac{T}{T_0} + C_8 \frac{S}{S_0} + C_9 \left(\frac{T}{T_0}\right)^2$ (5.18)

with values T_0, S_0, ρ_0 and atmospheric pressure as above C_6 = 0,97529, C_7= - 0,00006, C_8 = 0,02737, C_9 = - 0,0014.



SEAWATER STATE EQUATION

If the compression of the liquid, i.e. the variation of density and the pressure ratio is taken into account, there are:

$$\frac{\rho}{\rho_0} = \left[C_6 + C_7\frac{T}{T_0} + C_8\frac{S}{S_0} + C_9\left(\frac{T}{T_0}\right)^2 + C_{10}\frac{T}{T_0}\frac{S}{S_0}\right] \times \left[1 + \frac{P-P_0}{P_0}C_{11}\right] \quad (5.19)$$

Where P_0 is pressure is equal to 1 at., C_{11} is a constant quantity and a compression-resistant coefficient, can take: C_{10} = - 0,00119, C_{11} = 0,428.10^{-4}.

If you view the pressure as proportional to the depth, the equation (5.19) has:

$$\frac{\rho}{\rho_0} = \left[C_6 + C_7\frac{T}{T_0} + C_8\frac{S}{S_0} + C_9\left(\frac{T}{T_0}\right)^2 + C_{10}\frac{T}{T_0}\frac{S}{S_0}\right] \times \left[1 + C_{12}\frac{Z}{Z_0}\right] \quad (5.20)$$

And simpler dependency(5.20) is:

$$\frac{\rho}{\rho_0} = \left[C_6 + C_7\frac{T}{T_0} + C_8\frac{S}{S_0} + C_9\left(\frac{T}{T_0}\right)^2 + C_{10}\frac{T}{T_0}\frac{S}{S_0}\right] + C_{13}\frac{Z}{Z_0} \quad (5.21)$$

Where Z0 = 1km, it is possible to take C12 = 0.00428, C13 = 0.0043; (1.44) and (1.45) by Linheikin, Mamaev, Vaxilev. Equations (5.19) and (5.20), although not allowing for accurate calculation of density, have been used to solve most of the problems of sea current theory that involve examining the nonlinear interactions of the fields of flow velocity, density, temperature, and salt level.

TURBULENT MOTION, REYNOLDS STRESS

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The system of moving equations, continuous, state and preservation of salt heat is closed. But to get accurate results from solving those equations is impossible because the movement of the real liquid always has a tangled feature. Therefore, we must consider the tangled characteristics in these equations. Performs real motion in the form of medium motion and sublimation motion:

$$\mathrm{u} = \bar{u} + \mathrm{u}'; \mathrm{v} = \bar{v} + \mathrm{v}'; \mathrm{w} = \bar{w} + \mathrm{w}'$$

$$\mathrm{P} = \bar{P} + \mathrm{P}'; \mathrm{T} = \bar{T} + \mathrm{T}'; \mathrm{s} = \bar{s} + \mathrm{s}'; \rho = \bar{\rho} + \rho$$

Reynolds' 5 conditions for any function:

$$\mathrm{q}_1 + \mathrm{q}_2 = \bar{\mathrm{q}}_1 + \bar{\mathrm{q}}_2 \qquad (5.22)$$

$$\overline{\mathrm{aq}_1} = \overline{\mathrm{aq}}, \mathrm{khi\ a} = \mathrm{const} \qquad (5.23)$$

$$\bar{\mathrm{a}} = \mathrm{a}, \mathrm{khia} = \mathrm{const} \qquad (5.24)$$

$$\frac{\overline{\partial q_1}}{\partial x_i} = \frac{\partial \overline{q_1}}{\partial x_i}; \frac{\overline{\partial q_1}}{\partial t} = \frac{\partial \overline{q_1}}{\partial t}; \qquad (5.25)$$

$$\overline{\overline{\mathrm{q}_1} \cdot \mathrm{q}_2} = \overline{\mathrm{q}_1} \cdot \overline{\mathrm{q}_2} \qquad (5.26)$$

$\mathrm{q}_1 = \bar{q}_1 + \mathrm{q}'_1$; $\mathrm{q}_2 = \bar{q}_2 + \mathrm{q}'_2$;

$$\overline{\bar{\mathrm{q}}_1} = \bar{\mathrm{q}}_1 \qquad (5.27)$$

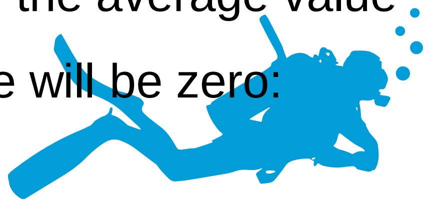
$$\overline{\mathrm{q}'_1} = 0 \qquad (5.28)$$

$$\overline{\bar{\mathrm{q}}_1 \cdot \overline{\mathrm{q}_2}} = \overline{\mathrm{q}_1} \cdot \overline{\mathrm{q}_2} \qquad (5.29)$$

$$\overline{\overline{\mathrm{q}_1} \cdot \mathrm{q}'_2} = 0 \qquad (5.30)$$

When considering the velocity field we have the average value of speed pulses over a certain period of time will be zero:

$$\overline{\mathrm{u}'} = \frac{1}{\mathrm{T}} \int_{\mathrm{t}}^{\mathrm{t+T}} \mathrm{u}'\mathrm{dt} = 0 \qquad (5.31)$$



TURBULENT MOTION, REYNOLDS STRESS

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MOTION EQUATION IN TURBULENT MOTION

At any time and points, the speed components must satisfy the Navie-Stoc equation, so according to the Ox axis we have:

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial u'}{\partial t}+(\bar{u}+u')\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial u'}{\partial x}\right)+(\bar{v}+v')\left(\frac{\partial \bar{u}}{\partial y}+\frac{\partial u'}{\partial y}\right)+(w'+\bar{w})\left(\frac{\partial \bar{u}}{\partial z}+\frac{\partial u'}{\partial z}\right)=X-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}-\frac{1}{\rho}\frac{\partial P'}{\partial x}+\upsilon\Delta\bar{u}+\upsilon\Delta u' \quad (5.32)$$

According to the Reynolds systems and the consequences we have: $\overline{u'}=0;\ \overline{\frac{\partial u'}{\partial x}}=0;\ \overline{\frac{\partial u'}{\partial t}}=0 \qquad (5.33)$

When we take the equation average (5.32) over the T period, we have:

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}+\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}=X-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}+\upsilon\Delta\bar{u} \qquad (5.34)$$

Hay: $\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}=X-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}-v\Delta\bar{u}-\frac{\overline{u'\partial u'}}{\partial x}-\frac{\overline{v'\hat{u}'}}{\partial y}-\frac{\overline{w'\partial u'}}{\partial z}=X-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}+v\overline{\Delta u}-\frac{\overline{u'^2}}{\partial x}-\frac{\overline{\partial u'v'}}{\partial y}-\frac{\overline{\partial u'w'}}{\partial z}+u'\frac{\partial u'}{\partial x}+$ $u'\frac{\partial v'}{\partial y}+u'\frac{\partial w'}{\partial z}$

When using continuous equations for medium and pulse motion and viewing the liquid as uncompressed, we have:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Therefore: $\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}=X-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}=\upsilon\Delta\bar{u}-\frac{\overline{\partial u'^2}}{\partial x}-\frac{\overline{\partial u'v'}}{\partial y}-\frac{\overline{\partial u'w'}}{\partial z}$

$$X-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}+\frac{1}{\rho}\left(\varepsilon\Delta\bar{u}-\rho\frac{\overline{\partial u'^2}}{\partial x}-\rho\frac{\overline{\partial u'v'}}{\partial y}-\rho\frac{\overline{\partial u'w'}}{\partial z}\right) \qquad (5.35)$$

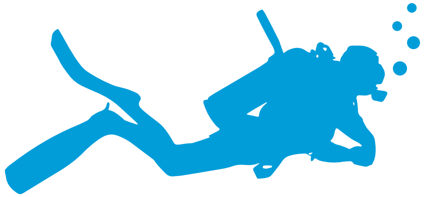
MOTION EQUATION IN TURBULENT MOTION

Thus the Navie-Stoc equation with tangled motion has been taken on average different from the previous equation of having additional nonlinear components.:

$$-\frac{\overline{\partial(u'^2)}}{\partial x}-\frac{\overline{\partial(u'v')}}{\partial y}-\frac{\overline{\partial(u'w')}}{\partial z}$$ theo trục Ox;

$$-\frac{\overline{\partial(u'v')}}{\partial x}-\frac{\overline{\partial(v'^2)}}{\partial y}-\frac{\overline{\partial(u'w')}}{\partial z}$$ theo trục Oy;

$$-\frac{\overline{\partial(u'w')}}{\partial x}-\frac{\overline{\partial(v'w')}}{\partial y}-\frac{\overline{\partial(w'^2)}}{\partial z}$$ theo trục Oz;



TURBULENT MOTION, REYNOLDS STRESS

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Motion Equation In Turbulent Motion

Similar to the viscous case of the element viscosity we have:

$$-\overline{\rho u'^2} = 2\varepsilon \frac{\partial \bar{u}}{\partial x} = \tau_{xx}$$

$$-\overline{\rho u'v'} = \overline{\rho v'u'} = \varepsilon\left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right) = \tau_{yx} = \tau_{xy}$$

$$-\overline{\rho u'w'} = \overline{\rho w'u'} = \varepsilon\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right) = \tau_{zx} = \tau_{xz} \qquad (5.36)$$

Với $\varepsilon = \upsilon\rho$

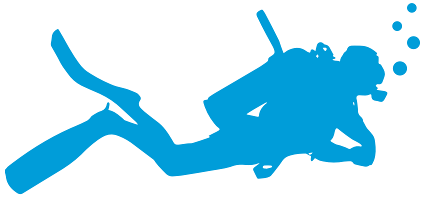
$$-\overline{\rho v'^2} = 2\varepsilon \frac{\partial \bar{v}}{\partial y} = \tau_{yy}$$

$$-\overline{\rho v'w'} = \overline{\rho w'v'} = \varepsilon\left(\frac{\partial v}{\partial z} + \frac{\partial \bar{w}}{\partial y}\right) = \tau_{zy} = \tau_{yz}$$

$$-\overline{\rho w'^2} = 2\varepsilon \frac{\partial \bar{w}}{\partial z} = \tau_{zz}$$

It is the component of reynolds tenxo or tangled tenxo of tangled motion:

$$\begin{pmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix}$$



TURBULENT MOTION, REYNOLDS STRESS

Motion equation in turbulent motion

If you consider flat movement, the movement is constant. $\frac{du}{dt} = 0$ have a speed (u) parallel to the Ox axis and depend only on z, there are:

$$0 = X - \frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} + \frac{1}{\partial}\left(\varepsilon\frac{d^2\bar{u}}{\partial z^2} - \rho\frac{d\overline{u'w'}}{dz}\right)$$

$$0 = X - \frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} + \frac{1}{\partial}\frac{d}{dz}\left(\varepsilon\frac{d\bar{u}}{\partial z} - \rho\bar{u}'w'\right) \qquad (5.37)$$

Similar to element friction, to find the turbulent viscosity coefficient μ we write the pressure in the form of:

$$F = \mu\frac{d\bar{u}}{dZ} = \varepsilon\frac{d\bar{u}}{dz} - \rho\overline{u'w'}$$

Because in **turbulent** motion. $\varepsilon\frac{d\bar{u}}{dz}$ very small compared to $\rho\overline{u'w'}$ So we can skip and watch closely: $F = \mu\frac{d\bar{u}}{dz} = -\rho\overline{u'w'}$

Therefore $$\mu = -\rho\frac{\overline{u'w'}}{\left(\frac{d\bar{u}}{dx}\right)} \qquad (5.38)$$

It's a formula for calculating a turbulent viscosity coefficient. Theo Businesq the coefficient μ It depends mainly on the intensity of the disorder. The turbulent viscosity coefficient has a range from 101 - 103 CGS.

TURBULENT MOTION, REYNOLDS STRESS

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Motion equation in turbulent motion

The assertion of u'v', u'w', v'w' in the other mean line does not allow to say that there must be some correlation between the ascension u', v', w’. When there is no correlation, u'v’ is Zero. The correlation coefficient is calculated as follows:

$$K = \frac{\overline{u'v'}}{\sqrt{\overline{u'^2}}.\sqrt{\overline{v'^2}}} \qquad (5.39)$$

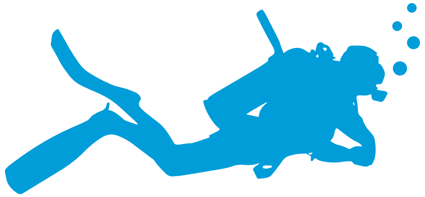
It has been determined that when the density is highly stable, the slime coefficient is small. So when the water mass has great stability, the Reynolds stress contains the w' component that must be smaller than reynolds stress without w', so the horizontal and vertical viscosity coefficients are different.

$$\tau_{xx} = 2\mu_V \frac{\partial \bar{u}}{\partial x} = -\rho\overline{u'^2} \qquad \tau_{zz} = 2\mu_h \frac{\partial \bar{w}}{\partial z} = -\rho\overline{w'^2} \qquad \tau_{xz} = \tau_{zx} = \mu_h \frac{\partial \bar{u}}{\partial z} + \mu_V = -\rho\overline{u'v'}$$

$$\tau_{yy} = 2\mu_V \frac{\partial \bar{v}}{\partial y} = -\rho\overline{v'^2} \qquad \tau_{xy} = \tau_{yx} = \mu_V\left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right) = -\rho\overline{u'v'} \qquad \tau_{xy} = \tau_{yx} = \mu_V\left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right) = -\rho\overline{u'v'} \qquad (5.40)$$

- $\mu_{h:}$ the turbulent viscosity coefficient of the horizontal speed gradient

- μ_v the turbulent viscosity coefficient of the vertical speed gradient.



TURBULENT MOTION, REYNOLDS STRESS

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Motion equation in turbulent motion

If the pressure pressures act on a fluid volume factor. $\delta x,\ \delta y,\ \delta z$, with hypothesis μ_h và μ_v is constant and considering the continuity equation for average motion, there are:

$$\delta x\delta y\delta z\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)=\delta x\delta y\delta z\left(\mu_h\frac{\delta^2\bar{u}}{\partial x^2}+\mu_h\frac{\partial^2\bar{u}}{\partial y^2}+\mu_h\frac{\partial^2\bar{u}}{\partial z^2}\right)$$

The average motion equation is rewritten as:

$$\frac{d\bar{u}}{dt}=\frac{\partial\bar{u}}{\partial t}+u\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}=X-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}+\frac{\mu_v}{\rho}\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}\right)+\frac{\mu_h}{\rho}\frac{\partial^2\bar{u}}{\partial z^2}$$

$$\frac{d\bar{v}}{dt}=\frac{\partial\bar{v}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial x}+\bar{w}\frac{\partial\bar{v}}{\partial z}=Y-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial y}+\frac{\mu_v}{\rho}\left(\frac{\partial^2\bar{v}}{\partial x^2}+\frac{\partial^2\bar{v}}{\partial y^2}\right)+\frac{\mu_h}{\rho}\frac{\partial^2\bar{v}}{\partial z^2}$$

$$\frac{d\bar{w}}{dt}=\frac{\partial\bar{w}}{\partial t}+\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{v}\frac{\partial\bar{w}}{\partial y}+\bar{w}\frac{\partial\bar{w}}{\partial z}=X-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial z}+\frac{\mu_v}{\rho}\left(\frac{\partial^2\bar{w}}{\partial x^2}+\frac{\partial^2\bar{w}}{\partial y^2}\right)+\frac{\mu_h}{\rho}\frac{\partial^2\bar{w}}{\partial z^2}) \qquad (5.41)$$

Coefficients $\mu_h\mu_v$ determined from Reynolds stress. Prandtl gives Reynolds stress formula as follows:

$\tau_{zx}=-\rho\,\overline{\mu'w'}\left|\frac{d\bar{u}}{dz}\right|\frac{d\bar{u}}{dz}$ (5.42) L is a turbulent road and $\mu=\rho\ell^2\frac{d\bar{u}}{dz}$ (5.43)

Thus the coefficient of μ proportional to the cube of the turbulent distance.

TURBULENT MOTION, REYNOLDS STRESS

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Salt Conservation Equation in turbulent motion

We have: $$\left[\frac{\partial}{\partial t}+(\bar{u}+u')\frac{\partial}{\partial x}+(\bar{v}+v')\frac{\partial}{\partial y}+(\bar{w}+w')\frac{\partial}{\partial t}\right](\bar{S}+S')=0 \qquad (5.44)$$

If the liquid to be uncompressed,: $$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

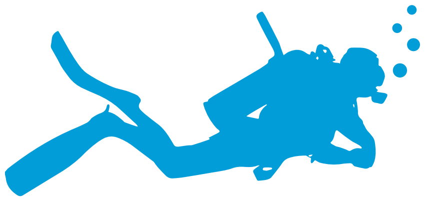
And take the average equation (5.44) we have: $$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}+\bar{v}\frac{\partial}{\partial t}+\bar{w}\frac{\partial}{\partial t}\right)\bar{S}+\frac{\partial}{\partial x}\overline{u'S'}+\frac{\partial}{\partial u}\overline{v'S'}+\frac{\partial}{\partial z}\overline{w'S'}=0 \qquad (5.45)$$

In the concept of a turbulent diffusion coefficient: Ax, Ay, Az of salt in the directions Ox, Oy, Oz. These coefficients are calculated according to expressions: $\overline{u's'}=-\frac{A_x}{\rho}\frac{\partial\bar{S}}{\partial x};\overline{v's'}=-\frac{A_y}{\rho}\frac{\partial\bar{S}}{\partial y};\overline{w's'}=-\frac{A_z}{\rho}\frac{\partial\bar{S}}{\hat{\partial} z}$ (5.46)

Hence, . $$\frac{\partial S}{\partial t}+\bar{u}\frac{\partial S}{\partial x}+\bar{v}\frac{\partial S}{\partial y}+\bar{w}\frac{\partial S}{\partial z}=\frac{\partial}{\partial x}\left(\frac{A_x}{\rho}\frac{\partial S}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{A_y}{\rho}\frac{\partial S}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{A_z}{\rho}\frac{\partial S}{\partial z}\right) \qquad (5.47)$$

(5.47) is the equation that diffuses salt in the sea. Similarly, we also found a thermal diffusion equation.

$$\frac{\partial\bar{T}}{\partial t}+\bar{u}\frac{\partial\bar{T}}{\partial x}+\bar{v}\frac{\partial\bar{T}}{\partial y}+\bar{w}\frac{\partial\bar{T}}{\partial z}=\frac{\partial}{\partial x}\left(\frac{A_{Tx}}{\rho}\frac{\partial\bar{T}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{A_{Ty}}{\rho}\frac{\partial\bar{T}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{A_{Tz}}{\rho}\frac{\partial\bar{T}}{\partial z}\right) \qquad (5.48)$$



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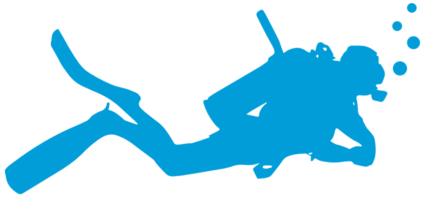
BOUNDARY CONDITIONS

Equations of motion are differential equations, in order to solve those equations, there must be boundary conditions or limit conditions. Boundary conditions are generally divided into three categories:

1. Dynamic boundary conditions.
2. Kinetic boundary conditions.
3. Thermal and salt conditions.



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Dynamic boundary conditions

These are conditions that indicate the continuity of the stressor at the boundary between the atmosphere and the ocean.

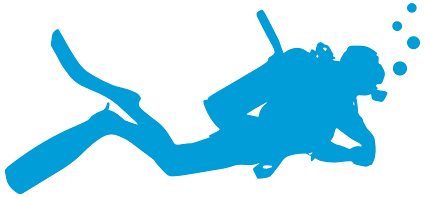
When z = - ζ (x,y,t), That is, on the free side of the ocean.:

Where Pa is the atmospheric pressure $\mathrm{P} = \mathrm{P_a}$ (5.49)

$$\mu_h \frac{\partial u}{\partial z} = -\tau_x;\ \mu_h \frac{\partial v}{\partial z} = -\tau_y \qquad (5.50)$$

where as тx, тy is a wind tangential stress on the sea surface.

Because of the lowering of ocean levels ζ It is usually very small compared to the depth of the sea, so these conditions are sometimes given on the non-noisy side of the sea.: z = 0.



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Kinetic boundary conditions

This condition indicates impermeability to liquids on the free side:

z = - ζ (x, y, t), tại đáy z = H (x,y) and in boundaries.

- Khi z= - ζ(x, y, t) $W = -\frac{d\xi}{dt} = -\left(\frac{d\xi}{dt} + u\frac{d\xi}{dx} + v\frac{d\xi}{dy}\right)$ (5.51)

- Khi z = H(x,y): Kinetic boundary condition can take two forms.:

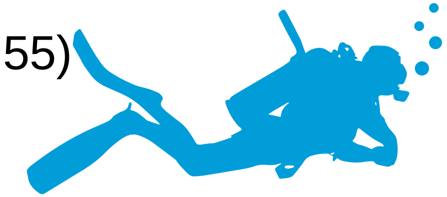
a) $W = u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}$ frictionless sliding conditions (5.52)

b) $u = v = 0, w = 0$ no-slip condition (5.53)

The choice of condition a) or b) is dependent on friction with the bottom. Sliding conditions do not take into account the bottom border layer.

- At the solid boundary: u = v = 0 adhesive and impermeable boundary conditions (5.54)

- At the liquid boundary, give before the distribution of the speed vector: $V_L = V_L(x, y, z)$ (5.55)



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Thermal and salt conditions

These conditions denote the effect of the transport of thermal and salt through the dividing sides.

On the free side of the ocean.: Khi z = - ζ(x,y,t) the general form of these conditions is:

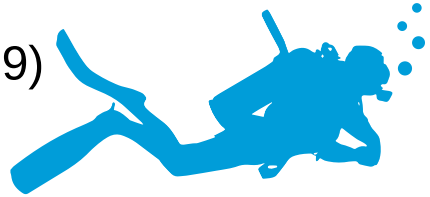
$$\gamma T + \delta \frac{\partial T}{\partial z} = G_T \qquad \gamma S + \delta \frac{\partial S}{\partial z} = G_S \qquad (5.56)$$

if δ = 0 The condition is for the values of the function itself., and if γ = 0 then for the gradient of that function.

- At the bottom and at the lateral solid boundaries, for conditions without the flow of thermal and salt according to the route method with the margin: $\frac{\partial T}{\partial n} = \frac{\partial S}{\partial n} = 0$ (5.57)
- At the side fluids boundaries: $\frac{\partial T}{\partial n} = G_{Tn}; \frac{\partial S}{\partial n} = G_{Sn}$ (5.58)

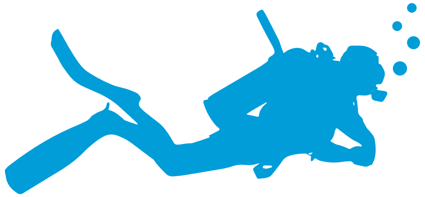
The initial conditions reflect the thermal state of the ocean at the initial time t = 0. It is usually required to give in advance the field of ocean features at the initial time:

$$u = u^{(0)}, v = v^{(0)}, w = w^{(0)}, p = p^{(0)}, T = T^{(0)}, s = s^{(0)}, \rho = \rho^{(0)} \qquad (5.59)$$



Classification of non-stop processes in the ocean and some approximations applied to currents research

When studying the ocean, there are seeing the phenomenon of fluctuations over time of marine fields such as the velocity field., temperature field T^0, salt degree field $S^0/_{00}$, density ρ..., they make up a variety of physical processes in the ocean. To classify these processes in time and space, similar to the classification of changes in climate fields, one derives from the spectrum of the cycle, which divides them into seven time periods.



CLASSIFICATION OF NON-STOP PROCESSES

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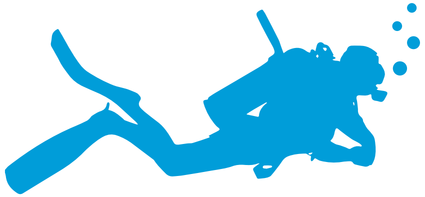
1. Small-scale phenomena: Cycles from a few seconds to tens of minutes.
2. Medium-scale phenomena: Cycles from a few hours to daily.
3. Syn scale change: Cycles a few days to months
4. Season fluctuations: Five-year cycles and larger.
5. Changes between years: I.e. changes consistent with the state of large seas and of the entire atmosphere from year to year.
6. Changes in the century: Cycles of several decades. It is the study of the connection between the ocean and changes in the century of climate. \
7. Changes between centuries: Cycles of hundreds of years and larger. It is the study of the connection between the ocean and the fluctuations between the centuries of climate.

SOME APPROXIMATIONS APPLIED TO CURRENTS RESEARCH

For average movements, the following approximations are correct.:

1. **QUASI-STATIC APPROXIMATION**
2. **APPROX. BUSINESQ**
3. **APPROXIMATIONS TO THE CORIOLIS FORCE**
4. **GEOLOCATION SYSTE**



QUASI-STATIC APPROXIMATION

The studies of medium and large-scale processes in the ocean (vertical scale H ≈ 100 m ÷ 1 km and horizontal scale (L ≈ 100 ÷ 1000 km) show that vertical velocity is much smaller than horizontal velocity. Consider the order of quantity in the conservation of mass equation (the continuity equation):

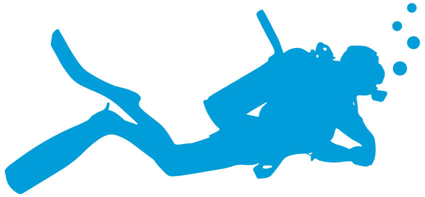
$$\mathrm{W} = \mathrm{H.U/L} \quad \text{suy ra } \mathrm{W} = 10^{-3}\mathrm{U} \qquad (5.60)$$

Where W, U are the characteristic quantities of the vertical and horizontal speeds.

Since the vertical velocity in the ocean is very small, it is possible to write the equation of vertical motion as

$$\frac{\partial P}{\partial z} = g \cdot \rho \qquad (5.61)$$

like the static equation.



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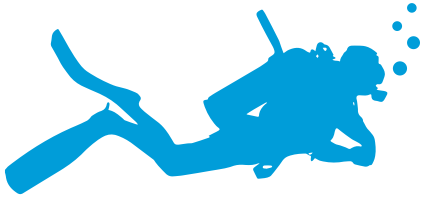
APPROX. BUSINESQ

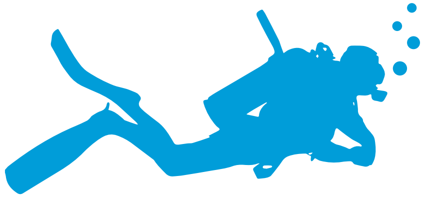
We know that the density of water in the ocean changes very little : $\frac{\partial\rho}{\rho} \approx 10^{-3}(\delta\rho$ is the density anomaly), so density ρ can be replaced by ρ_0 (average density), then the equation for conservation of mass is written as: $\overrightarrow{\mathrm{divV}} = 0$ (5.62)

(incompressible condition of seawater).



APPROXIMATIONS TO THE CORIOLIS FORCE

When studying medium- and large-scale motion in the ocean as known /W/ <</U/, the term with coefficient 2ωwcosϕ in the component of the Coriolis force along the Ox axis can be ignored. But it may be necessary to account for this term in the narrow band at the equator.



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LECTURE

MODELLING THE MARINE ENVIRONMENT

Lecturer: Prof. Nguyen Ky Phung

MSc. Dang Thi Thanh Le

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Lecture 6

THEORY OF CURRENTS

Lecturer: Professor. Nguyen Ky Phung
Ms. Dang Thi Thanh Le

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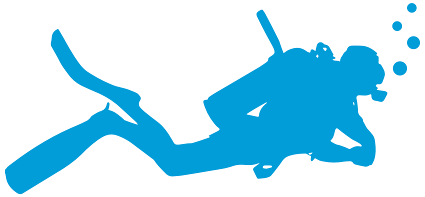


GEOSPHERE CURRENTS

❖ INCLINATION OF ISOBARIC SURFACES IN FLUIDS

In the conditions that (1) frictionless horizontal flow at a constant speed, (2) the single external force (gravity), and (3) no vertical movement, the horizontal components of Coriolis force and gradient pressure are balanced:

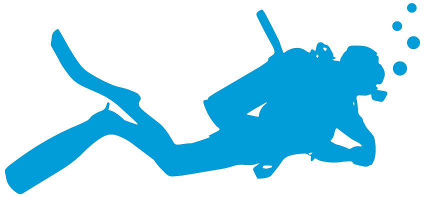
$$2\omega \, sin \, \varphi \, v = \frac{1}{\rho}\frac{\partial P}{\partial x} \qquad (6.1)$$

$$2\omega \sin \varphi \, u = -\frac{1}{\rho}\frac{\partial P}{\partial y}.$$

If we take the cubes of each equation and add them together, we have:

$$\frac{\partial P}{\partial n} = 2\omega\rho c \sin \varphi \qquad (6.2)$$

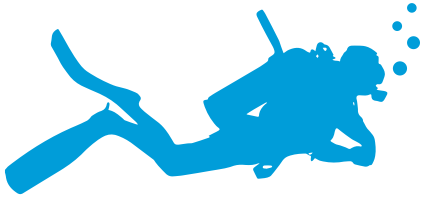
Ở đây $c = (u^2 + v^2)^{1/2}, \frac{\partial P}{\partial n} = \left[\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2\right]^{1/2}$



GEOSPHERE CURRENTS

❖ INCLINATION OF ISOBARIC SURFACES

Equation (6.2) shows the requirement for the forces balance is the equalization between the Coriolis force and horizontal pressure. Thus, the horizontal flow vector is parallel to the broken isobaric lines in a direction that the larger isobaric lines in the Northern Hemisphere have located on the right in the direction of the flow and vice versa in the Southern Hemisphere. This type of flow is called the ***barotropic flow*** and the balance of forces represented by equation (6.2) is called the ***barotropic flow*** equilibrium.



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GEOSPHERE CURRENTS

❖ INCLINATION OF ISOBARIC SURFACES IN FLUIDS

Replace the horizontal pressure gradient in Equation (6.1) with the angle of inclination of the isobaric surfaces. Figure 6.1a shows the inclination of the isostatic surfaces relative to the equipotential surfaces. The nOz plane is perpendicular to the flow rate c. Pressure at point A is equal to “P” and at point B is equal to “$P + \Delta P = P + \rho g \Delta z$”, where ρ − water column density between points C and *B*

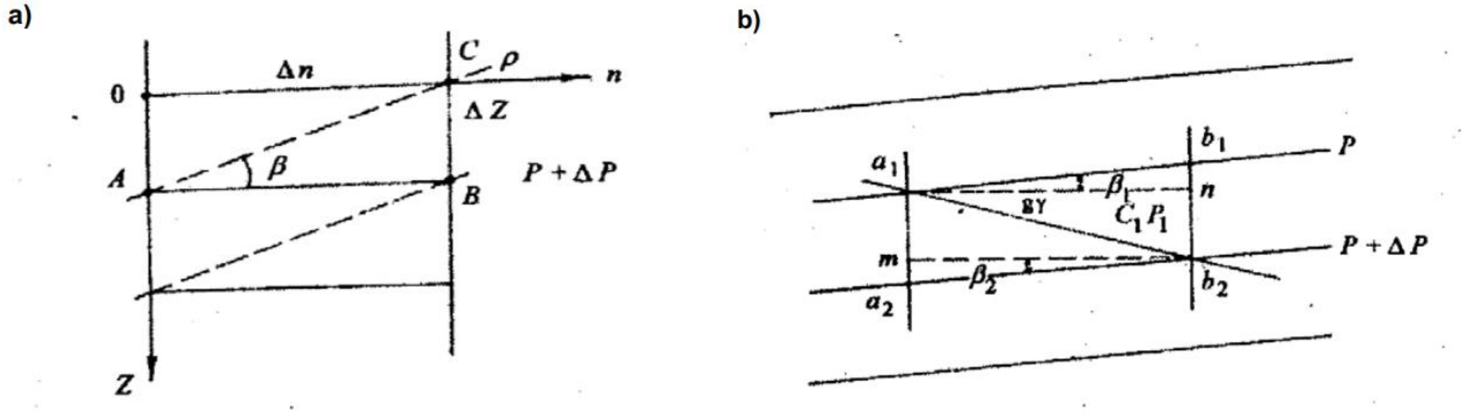
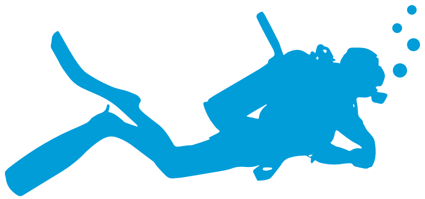


Figure 6.1. Inclination of isobaric surfaces



GEOSPHERE CURRENTS

❖ Inclination OF ISOBARIC SURFACES

Thus:

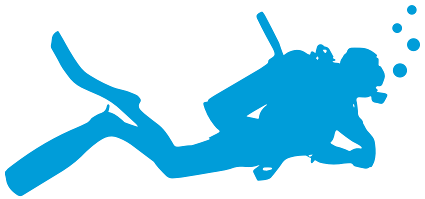
$$\frac{\Delta P}{\Delta n} = \rho g \frac{\Delta z}{\Delta n}$$

$$\frac{\partial P}{\partial n} = \rho g t\, g\, \beta \qquad (6.3)$$

If the Oz axis trend downwards, the angle β will follow clockwise rotation. From equations (6.2) and (6.3), the tg values β express as:

$$\mathrm{tg}\,\beta = \frac{2\omega c \sin\varphi}{g}$$

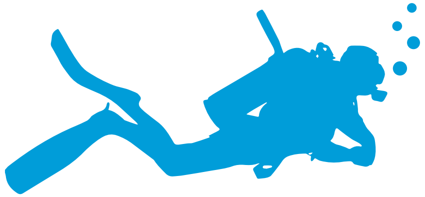
From those equations, the angle of inclination of an isobaric surface is proportional to the flow rate at the depth of that surface.



GEOSPHERE CURRENTS

❖ INCLINATION OF ISOBARIC SURFACES

The impact of the Coriolis force in actual currents creates the horizontal Circulation trend. As a result, lighter water in the upper layer moves on the right side of the flow direction and vice versa while the Southern Hemisphere sees the opposite movement. Therefore, with the same inclination of isobaric surfaces, the inclination of isopycnic surfaces appears. The angle between isobaric surfaces and isobaric volume is contrary. Moreover, the inclination of isopycnic surfaces appears somehow leads to the inclination of isobaric surfaces which causes horizontal gradient pressure. The distribution of isopycnic surfaces can relate to the water movement,



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GEOSPHERE CURRENTS

❖ INCLINATION OF ISOBARIC SURFACES IN FLUIDS

Figure 6.1b shows the inclination of the isopycnic curves relative to the isostatic surfaces. Since the pressures at points a2 and b2 are equal, then: :

$$g\rho_1(b_1b_2) = g\rho_2(a_1a_2) \qquad (6.5)$$

Besides,

$$b_1b_2 = b_1n + b_2n \qquad a_1a_2 = a_1m + a_2m \qquad \frac{b_2n}{a_1n} = \operatorname{tg}\gamma$$

$$\frac{b_1n}{a_1n} = \operatorname{tg}\beta_1 \qquad \frac{a_2m}{b_2m} = \operatorname{tg}\beta_2$$

Ta change of expression (6.5), simplified g : $\rho_1(a_1n\operatorname{tg}\beta_1 + a_1n\operatorname{tg}\gamma) = \rho_2(a_1n\operatorname{tg}\gamma + a_1n\operatorname{tg}\beta_2)$ (6.6)

simplified the number of term a_1n and substitute (6.6) the values: $t\,g\,\beta_1 = \dfrac{2\omega c_1 \sin\varphi}{g}$ và $t\,g\,\beta_2 = \dfrac{2\omega c_2 \sin\varphi}{g}$

$$\operatorname{tg}\gamma = \frac{2\omega\sin\varphi}{g}\cdot\frac{\rho_1c_1 - \rho_2c_2}{\rho_2 - \rho_1} \qquad (6.7)$$

GEOSPHERE CURRENTS

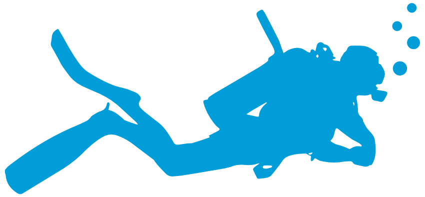
❖ INCLINATION OF ISOBARIC SURFACES IN FLUIDS

In meteorology, this similar to the Margules formula which is:

1) The position of the isopycnic lines on the cross-section allows consideration of the presence of flow perpendicular to the plane of the section and its direction;

2) The greater the inclination of the isopycnic lines, the smaller the density difference of the layers, the larger the speed difference. In immobilized layers, the isopycnic curves as well as the isobaric lines are horizontal

3) If layers of the same density move at different velocities, then γ = 90^0. In this case, the layers are very unstable and the γ has no meaning.



GEOSPHERE CURRENTS

❖ METHOD OF DYNAMIC CALCULATION OF GEOSPHERE CURRENTS

From the equations (6.1), find equation for u and v:

$$u = -\frac{1}{2\omega\rho\sin\varphi}\cdot\frac{\partial P}{\partial y} = -\frac{\alpha}{2\omega\sin\varphi}\cdot\frac{\partial P}{\partial y}$$

$$v = \frac{1}{2\omega\rho\sin\varphi}\cdot\frac{\partial P}{\partial x} = \frac{\alpha}{2\omega\sin\varphi}\cdot\frac{\partial P}{\partial x} \qquad (6.8)$$

These equations represent the balance between the horizontal composition of the friction force and the Coriolis force produced by the movement itself.

Because: $\alpha\partial P = \partial D$, then the expressions (6.8) can be rewritten as follows: $\mathrm{U} = \frac{1}{2\omega\sin\varphi}\cdot\frac{\partial \mathrm{D}}{\partial \mathrm{y}}$ $V = \frac{1}{2\omega\sin\varphi}\cdot\frac{\partial \mathrm{D}}{\partial \mathrm{x}}$ (6.9)

If the direction of n is the greatest inclination of the isotropic side, then we have the speed:

$c = \frac{1}{2\omega\sin\varphi}\cdot\frac{\partial D}{\partial n}$ (6.10) Where $\frac{\partial D}{\partial \mathrm{n}}$ actual inclination of the isobaric surfaces relative to the equipotential plane

GEOSPHERE CURRENTS

❖ METHOD OF DYNAMIC CALCULATION OF GEOSPHERE CURRENTS

The relative "inclination" between two hydrological stations is not difficult to determine. Suppose we have two hydrological stations A and B . Consider two isostatic surfaces P_1 and P_2. Let the distance between A and B be ℓ. Then, for the isobaric surface P1, the flow rate in the direction perpendicular to AB is :

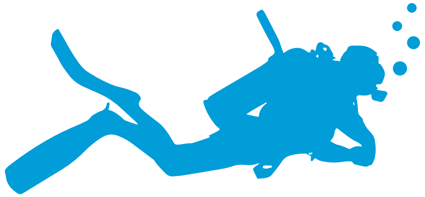
$$C_1 = \frac{D_{B_1} - D_{A_1}}{2\omega\ell \sin\varphi}$$

The line segment $AB = \ell$ is accepted as the differential factor dn , dynamic height D_{A_1} and D_{B_1} of the isobaric surface P1 relative to the equipotential surface is unknown at present. The speed at the isobaric surface P2 is determined by the same formula:

$$C_2 = \frac{D_{B_1} - D_{B_2}}{2\omega\ell \sin\varphi}$$

Take the first expression minus the second, we have results as below

$$C_1 - C_2 = \frac{\left(D_{B_1} - D_{B_2}\right) - \left(D_{A_1} - D_{A_2}\right)}{2\omega\ell \sin\varphi} = \frac{\Delta D_B - \Delta D_A}{2\omega\ell \sin\varphi} \qquad (6.11)$$

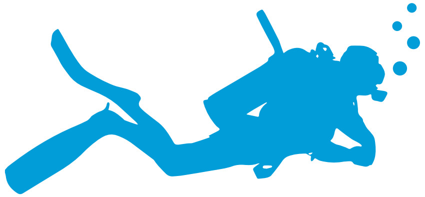


GEOSPHERE CURRENTS

❖ METHOD OF DYNAMIC CALCULATION OF GEOSPHERE CURRENTS

Thus, the dynamic method only allows for determining the difference in speeds. If we know the flow rate at a certain cross-section (where the flow velocity is zero), the problem is simply solved. But in practice, we almost always do not know that speed, so the problem is how to choose the zero surface. Thanks to equation (6.11), it is possible to calculate the real speed of the flow at different water levels.

Based on equation (6.10), it can be determined that the zero surface is the depth at which the horizontal gradient components of the dynamic depth approach 0.



GEOSPHERE CURRENTS

❖ METHOD OF DYNAMIC CALCULATION OF GEOSPHERE CURRENTS

The Defant method is the most common method for choosing the zero side, which is based entirely on the dynamic characteristics of the flow and does not contain assumptions like other methods.

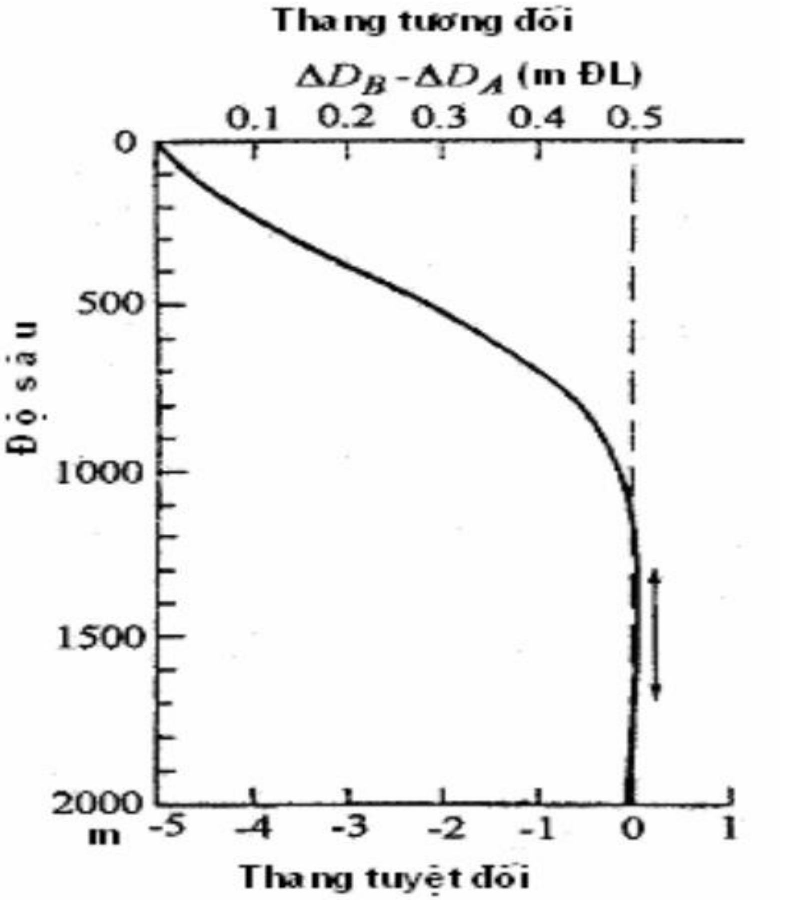


Figure 6.2. To identify the face without the Defant method

GEOSPHERE CURRENTS

❖ METHOD OF DYNAMIC CALCULATION OF GEOSPHERE CURRENTS

When searching for zero surfaces, Defant noticed that most of the differences in curves between dynamic depths in two oceanographic stations (Figure 6.2) for different station pairs are characterized by the existence of more or less straight segments. standing, for pairs of neighboring stations, they are distributed at approximately the same depths.

Within those segments, the dynamic depth differences remain constant. This means the speeds of the currents are identical.

If the 0 mark is not located in the vicinity of this vertical segment, then in the whole layers that have no difference in flow rates. If the 0 mark is not located far enough from this section, then the flow rates in the entire layer will be equally large. The latter is also less realistic, so Defant assumes that the flow rates throughout the layer are equal to the difference between the same dynamic depths, while the zero surface is in the center of the layer.

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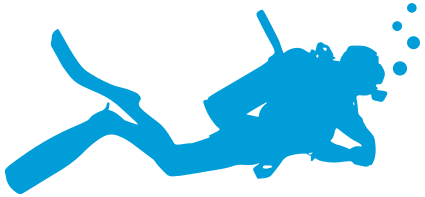
GEOSPHERE CURRENTS

❖ METHOD OF DYNAMIC CALCULATION OF GEOSPHERE CURRENTS

There is another method is use to select the zero surface based on the analysis of the individual volume difference curves between neighbouring stations – Parr's method. This method is attributed to the identification of variations in water layers between selected mass isomorphs and so on.

Since the fact that in the World Ocean does not exist on a single (continuous) side, instead of using a reference side as common, the geosphere currents are assumed to equal zero. For this purpose, the reference surface chosen in the class between 1000 and 2000 m is quite appropriate, while the surface at 3000 m of depth is chosen in some examples such as the southern Ocean

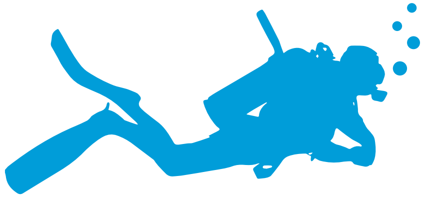


GEOSPHERE CURRENTS

❖ METHOD OF DYNAMIC CALCULATION OF GEOSPHERE CURRENTS

Other difficulties in determining the zero side, the method of dynamic calculation also has a series of disadvantages:

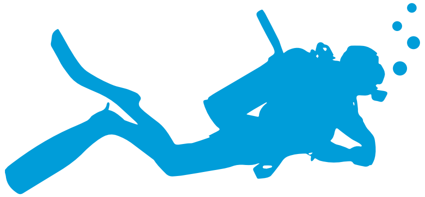
- As not mention to the pure flow component under the direct impact of tangential wind stress, exclusive the speed and direction of the wind, the swirling and non-stop components caused by forces not present in the basic equation (6.11), as well as ignoring the seabed topography.
- In addition, large deviations in flow rate can appear if the hydrological cross section is stimulated for a long period of time, furthermore not perpendicular to the direction of the flow, the distance between stations is not the same and quite large, especially in areas with front.



GEOSPHERE CURRENTS

❖ METHOD OF DYNAMIC CALCULATION OF GEOSPHERE CURRENTS

Despite such major drawbacks, the method of dynamic calculation due to simplicity and ease of use has been recognized worldwide and remains valid to this day. This method is often applied to standard cross sections, when performing standard cross sections always have to compare the results received with the estimated data in previous years. We also note that ocean circulation maps built on dynamic methods (Shott, 1933, Sverdrup, 1941, Ditrich, 1961, etc.) are generally quite consistent with observational data and overall ocean circulation mathematical modeling results.



GEOSPHERE CURRENTS

❖ DYNAMIC METHODS FOR CALCULATION OF GEOGRAPHIC FLUID

Figure 6.3 shows the dynamic surface map of Nam Duong as an example.

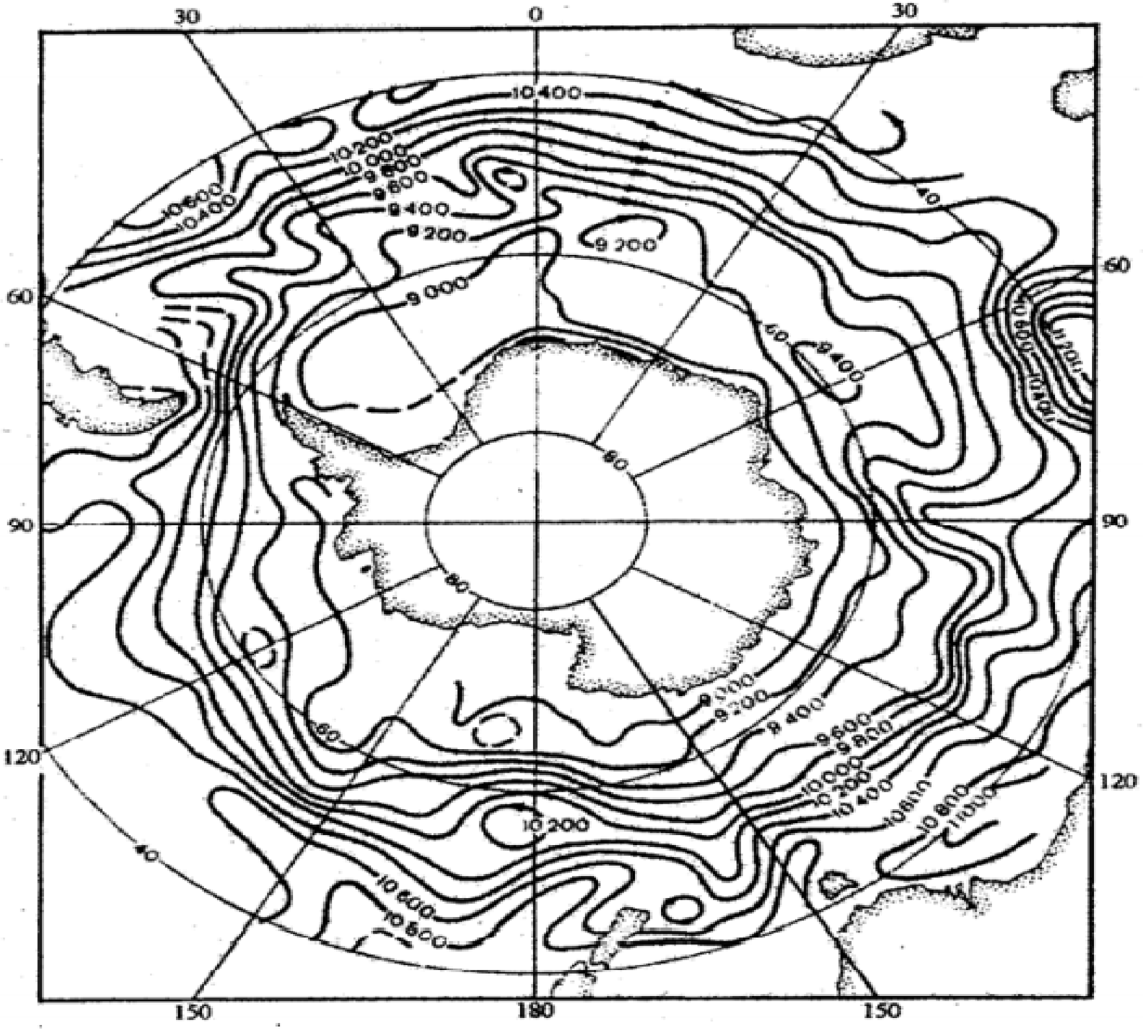


Figure 6.3: The dynamic surface map of Nam Duong as an example

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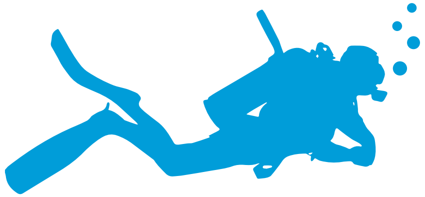
THEORY OF CURRENTS



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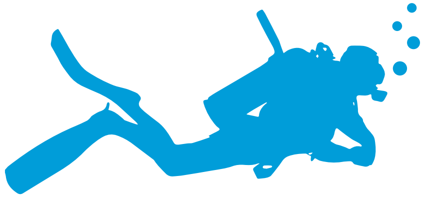
STEADY CURRENTS

❖ Ekman theory with deep currents

Since the friction stress of the wind is greater than the other forces that cause the flow, on average the wind flow contributes the largest part to the total speed of the flows, especially in the upper layer of the ocean. Ekman made the following assumptions:

1. The sea is shoreless and infinitely deep (to eliminate the effect of friction with the shore and bottom);
2. Wind and currents caused by it are stable and do not change over time;
3. The wind and current speed fields do not vary in the horizontal direction (no divergence);
4. The vertical component of the speed is absent because the motion occurs only in the horizontal direction and does not diverge;
5. Sea is homogeneous in density (to exclude density flow) and incompressible water;
6. The sea surface is the horizontal plane (to exclude the gradient component);
7. The accepted coefficient of tangle friction A_z remains constant with depth



STEADY CURRENTS

❖ Ekman theory with deep currents

With all the assumptions for steady flow, a turbulent frictional is the only force that transmits the impact of wind stress down to the depth and the Coriolis force is equal to it. The equation of motion in this case has the form:

$$\frac{A_z}{\rho}\frac{d^2u}{dz^2} + 2\omega v \sin\varphi = 0$$
$$\frac{A_z}{\rho}\frac{d^2v}{dz^2} - 2\omega u \sin\varphi = 0$$

Here we put:

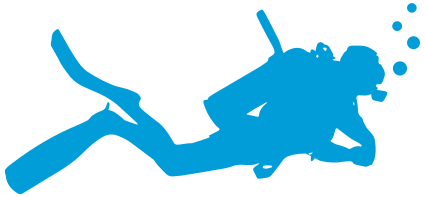
- The Y axis to coincide with the wind direction,
- The X axis is towards the right,
- The Z axis is pointing down

Transform the expressions above into forms:

$$\frac{d^2u}{dz^2} + \frac{2\rho}{A_z}\omega v \sin\varphi = 0$$
$$\frac{d^2v}{dz^2} - \frac{2\rho}{A_z}\omega \mathrm{u} \sin\varphi = 0 \qquad (6.12)$$

If symbolize: $\frac{\rho\omega\sin\varphi}{A_z} = a^2$ Then the equations (6.12) are rewritten to:

$$\frac{d^2u}{dz^2} + 2a^2v = 0$$
$$\frac{d^2v}{dz^2} - 2a^2u = 0 \qquad (6.13)$$



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STEADY CURRENTS

❖ Ekman theory with deep currents

This is a system of second-order ordinary differential equations and solutions of the form :

$$u = c_1 e^{az}\, cos(az + \phi_1) + c_2 e^{-az}\, cos(az + \phi_2)\,,$$
$$v = c_1 e^{az}\, sin(az + \phi_1) - c_2 e^{-az}\, sin(az + \phi_2)\,,$$

(c_1, c_2, ϕ_1, ϕ_2 - constants)

We state the first boundary condition: the flow rate when increasing depth needs to be limited, i.e.

$u \neq \infty$, $v \neq \infty$ khi $z \rightarrow \infty$

In this case, c1 must be zero, otherwise, the increasing of (z) the speed will increase infinitely. At the same time, it is no longer necessary to identify ϕ_1.

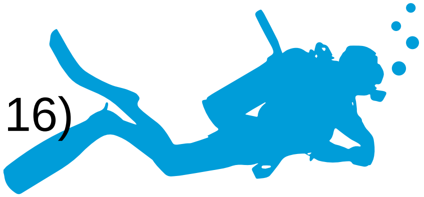
We rewrite the equations (6.14) as follows: $u = c_2 e^{-az} \cos(az + \phi_2)$; $v = -c_2 e^{-az} \sin(az + \phi_2)$ (6.15)

We set out the second boundary condition:

- At sea surface z = 0
- Wind tangeriial stress $\tau = A_z \dfrac{dc}{dz}$
- And axis Y in the direction gió.

Then, at z = 0 and the edge stress in the water just below the ocean surface will be equal to the wind gland friction, we have:

$$-A_z \frac{du}{dz} = 0$$
$$-A_z \frac{dv}{dz} = \tau \qquad (6.16)$$



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STEADY CURRENTS

❖ Ekman theory with deep currents

The speed module symbol at the surface is U_0, when:

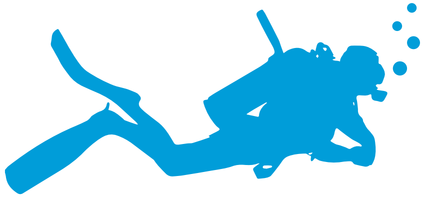
$$U_0 = \sqrt{u^2 + v^2} = \frac{\tau}{\sqrt{2}A_z a}. \qquad (6.17)$$

Substituting the value to equation (6.17), we get

$$U_0 = \sqrt{u^2 + v^2} = \frac{\tau}{\sqrt{2}A_z a}. \qquad (6.18)$$

Equation (6.17) can draw the conclusion that with the same conditions, the flow velocity decreases as latitude increases. Along with (6.17) the equations (6.15) can be rewritten

$$\begin{aligned} u &= U_0 e^{-az}\cos(45 - az) \\ v &= U_0 e^{-az}\sin(45 - az) \end{aligned} \qquad (6.19)$$



STEADY CURRENTS

❖ Ekman's theory with deep current

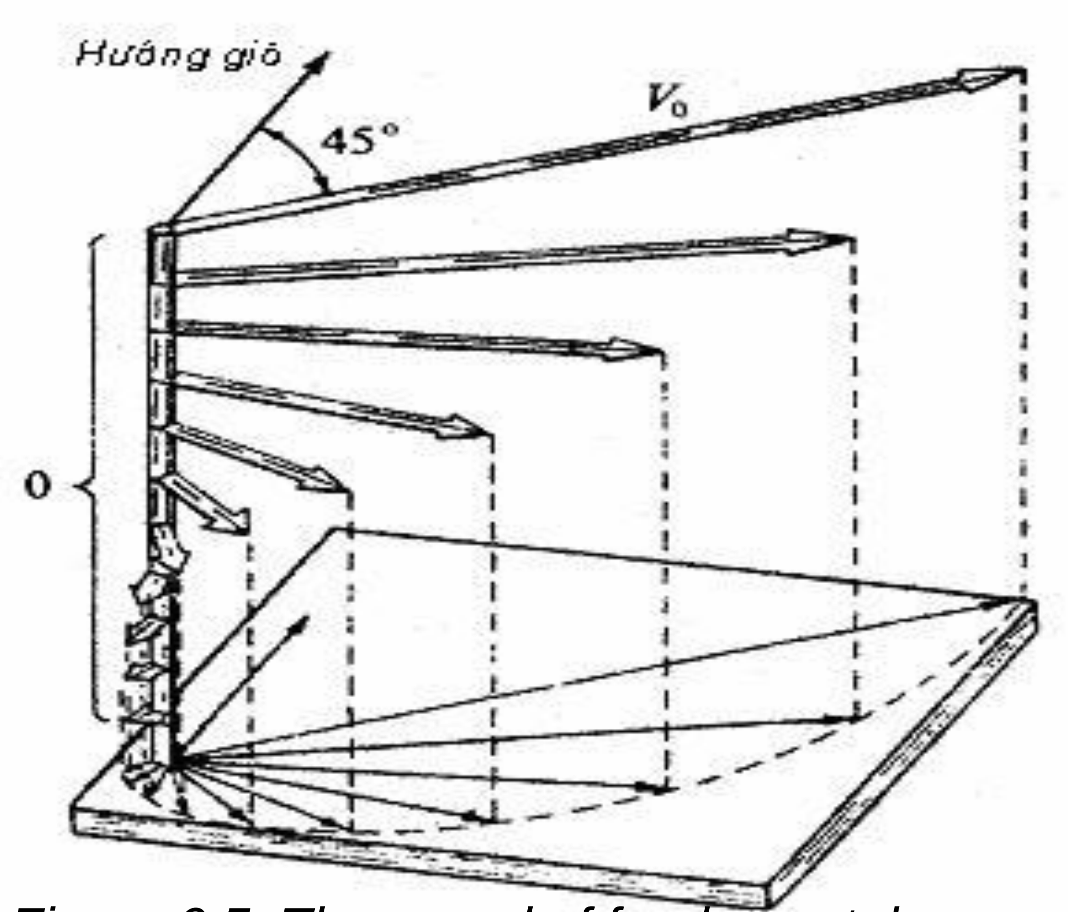


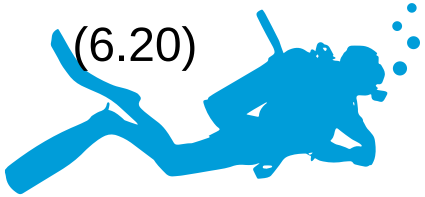
Figure 6.5. The speed of fundamental currents according to Ekman

Along with a decrease in speed with depth, the current turns to the right relative to its direction at sea level.

Figure 6.4 represents the speed line described in the shape of loga twist and represents a change in direction and the speed of wind flow in depth. Figure 6.4 shows that, at some depth, the speed vector will point in the opposite direction to the face flow.

It is often referred to as the depth of friction (rather the impact depth of friction) and the symbol in D.:

$$D = \frac{\pi}{a} = \pi\sqrt{\frac{A_z}{\rho\omega\sin\varphi}} \qquad (6.20)$$



STEADY CURRENTS

❖ Ekman theory with deep currents

The quantity Az is difficult to determine, so when there is flow monitoring data in the ocean surface layer, it is possible to find Az from formula (6.20) if the quantity D is known:

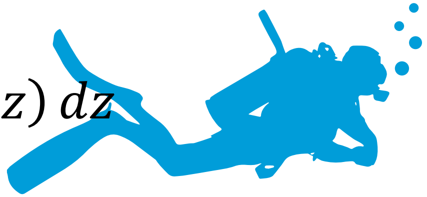
$$A_z = \frac{D^2 \rho \omega \sin \varphi}{\pi^2} \qquad (6.21)$$

The total flux of the drift is determined by integrating from zero to infinity in the coordinate axes directions:

$$S_x = \int_0^\infty u dz \text{ và } S_y = \int_0^\infty v dz \qquad (6.22)$$

Substitute u and v from (6.19) to (6.22):

$$S_x = U_0 \int_0^\infty e^{-az} \cos(45^\circ - az)\, dz \qquad S_y = U_0 \int_0^\infty e^{-az} \sin(45^\circ - az)\, dz$$



STEADY CURRENTS

❖ Ekman's theory for the deep sea

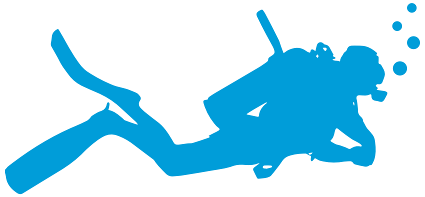
$$\int e^{az} \sin b\,x dx = \frac{e^{az}}{a^2+b^2}(a \sin bx - b \cos bx) \;,\; \int e^{az} \cos b\,x dx = \frac{e^{az}}{a^2+b^2}(a \cos bx - b \sin bx)$$

$$S_x \quad = \frac{U_0 e^{-az}}{2a^2}[-\mathrm{acos}(45-az) - \mathrm{asin}(45-az)]\Big|_0^\infty = \frac{U_0}{2a^2} a\sqrt{2} = \frac{U_0\sqrt{2}}{2a}$$

$$= U_0\sqrt{2}/2a \cdot \pi/a \cdot a/\pi = \frac{U_0\sqrt{2}D}{2\pi}$$

$$S_y = \frac{U_0 e^{-az}}{2a^2}[-a\sin(45-az) + a\cos(45-az)]\Big|_0^\infty = 0$$

$$S_x = \frac{U_0\sqrt{2}D}{2\pi}, \qquad S_y = 0 \qquad (6.23)$$



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STEADY CURRENTS

❖ Ekman's theory for the shallow sea

There is no difference in results for the shallow sea. By integral equation (6.13) and sets the additional conditions so that at the seabed both the speed components of u and v are equal to zero. Without repeating all of Ekman's arguments, we write:

$$u = Asha\xi \cos a\,\xi - Bcha\xi\, sin\, a\,\xi$$
$$v = Acha\xi \sin a\,\xi + Bs\mathrm{h}\, a\,\xi\, cos\, a\,\xi$$

ξ - vertical coordinates at the base.

The constants analyzed by A and B are equal to:

$$A = \frac{\tau D}{\pi A_z}\frac{\mathrm{ch}\, a\, d\cos ad + \mathrm{sh}\, a\, d\sin ad}{\mathrm{ch}\, 2\, ad + \mathrm{cos}\, a\, d}$$
$$B = \frac{\tau D}{\pi A_z}\frac{\mathrm{ch}\, a\, d\cos ad - sh\, ad\, \sin ad}{\mathrm{ch}\, 2\, ad + \mathrm{cos}\, a\, d} \qquad (6.24)$$

With d - is sea depth

STEADY CURRENTS

❖ Ekman's theory for the shallow sea

The angle between the flow direction at the surface and axis Y is determined by the expression:

$$\mathrm{tg}(U_0, Y) = \frac{U_0}{V_0} = \frac{\mathrm{sh}\,2ad - \sin 2ad}{\mathrm{sh}\,2ad + \sin 2ad} \quad (\mathbf{6.25})$$

where $2ad$ is the depth of sea $2ad = 2ad\,\frac{\pi}{a}\,\frac{a}{\pi} = \frac{2\pi d}{D}$

Then the quantity d/D can be consider as shallow water indicator

The table below shows the α value between the flow vector and the wind vector that depends on the quantity d/D

d/D	0.1	0.25	0.5	0.75	1	>1
α	5	21.5	45	45.5	45	45

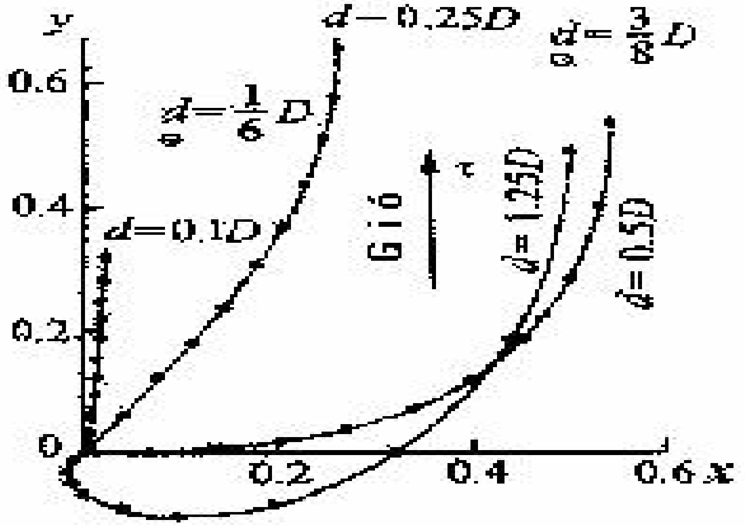


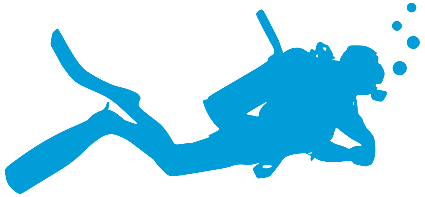
Figure 6.5 Current velocity heads in the sea of finite depth depending on the d/D ratio

From Figure 6.5 this infers that, when d > D the speed of the vectors the actual flow speed coincides with the case of the infinite deep sea (see Figure 6.4).

STEADY CURRENTS

❖ The development of fluency currents

Before stabilizing, the direction and velocity of flow may be much different than defined by formulas (6.19) and (6.24). Ekman looked at the development of drift currents in case the wind with constant intensity and direction began to effect the silent sea surface in a stable state. There is seeing that the flow at different water levels develops differently and the deeper you dive, the later stability appears.



STEADY CURRENTS

❖ The development of flowing currents

In Figure 6.6, the endpoint of the unstable flow vector draws a complex spiral-shaped curve that gradually approaches a stable value.

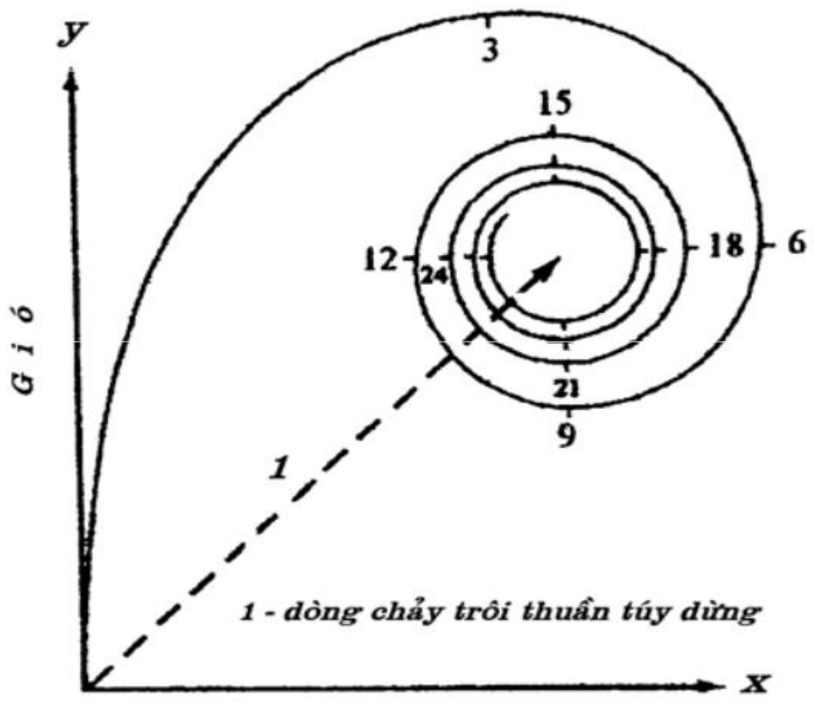
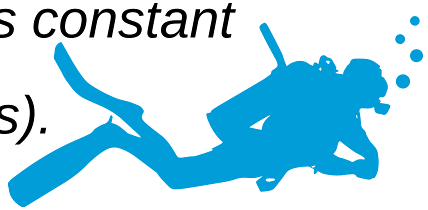


Figure 6.6. The velocity curve shows the development of pure drift current at the sea surface (time from wind arrival is constant equal to pendulum hours).



GRADIENT CURRENTS THEORY

In nature, the rise and fall of water surface occurs even far from shore. The inclination of the sea surface can create a pressure gradient that causes the gradient currents. Ekman made the following assumptions to simplify the process:

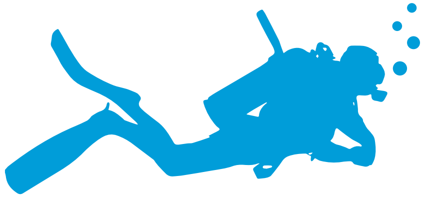
1) The landless sea and homogeneous in density;

2) The inclination of the sea surface is constant and stable in time and space;

3) Flat seabed;

4) Stable flow, no vertical components;

5) There is no fluctuation in the turbulent viscosity coefficient with depth.



GRADIENT CURRENTS THEORY

In this case, the following impact forces are the horizontal pressure, the Coriolis force, and the friction force in which the bottom friction is transmitted vertically, which constrains movement.

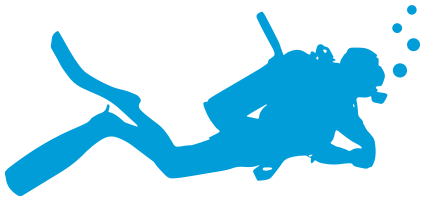
Motion equations write in the form of:

$$\frac{A_z}{\rho}\frac{d^2u}{dz^2} + 2\omega v \sin\varphi = 0 \qquad \frac{A_z}{\rho}\frac{d^2v}{dz^2} - 2\omega u \sin\varphi + g\sin\beta = 0 \qquad (6.26)$$

Formulas for determining gradien flow rate components are written as:

$$u = \frac{g\sin\beta}{2\omega\sin\varphi}\left[1 - \frac{\mathrm{ch}\, a(H+z)\cos a(H-z) + \mathrm{ch}\, a(H-z)\cos(H+z)}{\mathrm{ch}\, 2aH + \cos 2aH}\right] \qquad (6.27)$$

$$v = \frac{g\sin\beta}{2\omega\sin\varphi}\left[\frac{\mathrm{sh}\, a(H+z)\sin a(H-z) + sha(H-z)\sin(H+z)}{\mathrm{ch}\, 2aH + \cos 2aH}\right]$$



GRADIENT CURRENTS THEORY

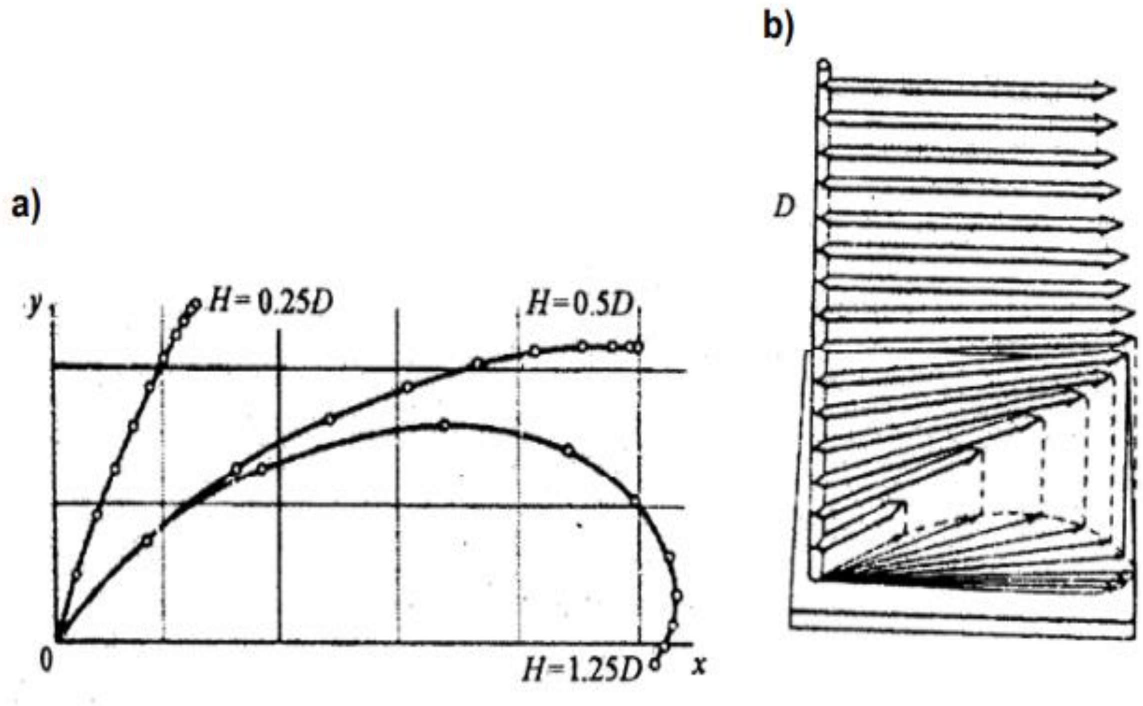
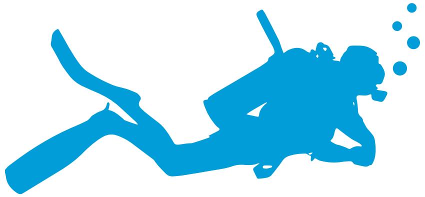


Figure 6.7.

Based on equations (6.27), (Figure 6.7a), the construction of the first curve's velocity current for three values of sea depth is expressed as a fraction of the depth friction. Figure 6.7 shows the stereoscopic change of gradient flow in different depths.



GRADIENT CURRENTS THEORY

At the seabed, the currents are zero according to the condition. As the increase in distance from the bottom, the flow velocity increases slowly turns to the right direction compared to the inclination of the water level. As the water depth large enough, maximum speed and deflection angle 90 ° achieved at a distance $D = \frac{\pi}{a}$ from the bottom. Since continuing leave from the bottom, the speed and direction of the flow remain constant until reaching the surface.

As such, the influence of bottom friction is spread upwards within the range layer of the D-thickness. Similar to the influence of the depth of friction in the Ekman drift, this layer is called the lower friction (the lower boundary of the depth influence the bottom friction).

The total gradient flux has components in both coordinate axes. The Y-axis composition is only significant in the layers near the bottom and when H > D it reaches a defined finite limit like the X-axis component:

$$S_y \to \frac{Dg\sin\beta}{4\pi\omega\sin\varphi}, S_x \to \frac{g\sin\beta}{2\omega\sin\varphi}\left(H - \frac{D}{2\pi}\right) \qquad (6.28)$$

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LECTURE

MODELLING THE MARINE ENVIRONMENT

Lecturer: Prof. Nguyen Ky Phung
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Lecture 7

PROCESS OF SUBSTANCE TRANSMISSION

Lecturer: Prof. Nguyen Ky Phung
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Lecture 7A

MASS BALANCING EQUATION

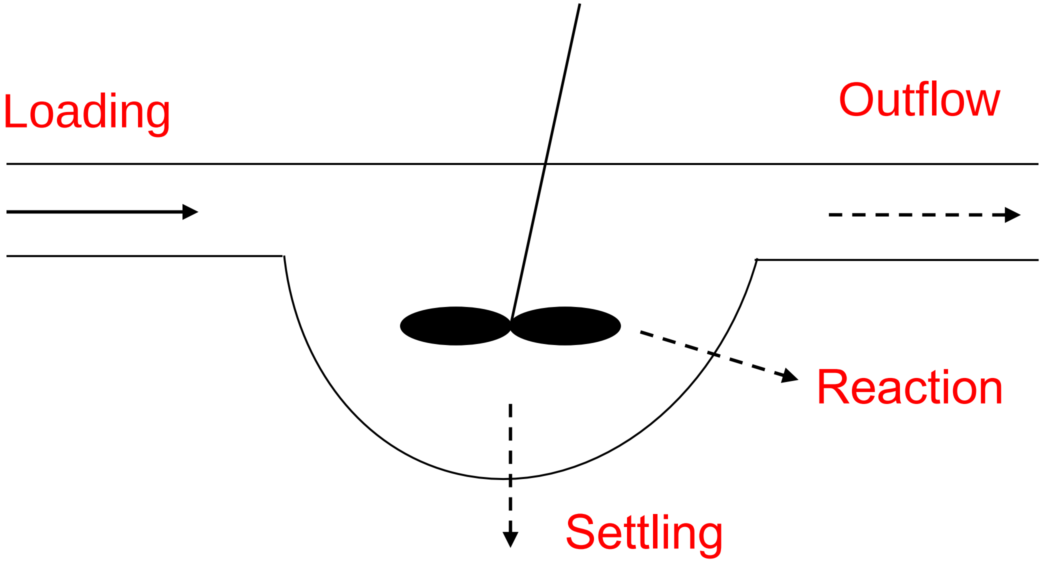
Cán Bộ Giảng Dạy: GS.TS. Nguyễn Kỳ Phùng
ThS. Đặng Thị Thanh Lê

CONTENTS

- **MASS balancing equation**
- **Steady - state system Solution**
  - Transfer functions
  - Residence
- **Unsteady - state system Solution**
  - Impulse loading
  - Step loading
  - Linear loading
  - Exponential loading
  - Sinusodial loading

MASS BALANCE FOR A WELL-MIXED LAKE

A completely mixing system. (CSTR – Continuously Stirred Tank Reactor) It's the simplest system that can be used to model a real water body.



For a finite time period, the mass balance of the system is described by the equation:

Accumulation = Loading - OutFlow - Reaction - Settling

MASS BALANCE FOR A WELL-MIXED LAKE (Cont)

- **Accumulation:** represent the change of Mass M of the system over a period of time t.

$$\left.\begin{aligned} \text{Accumulation} &= \frac{\Delta \mathrm{M}}{\Delta \mathrm{t}} \\ \mathrm{c} = \frac{\mathrm{M}}{\mathrm{V}} &\rightarrow \mathrm{M} = \mathrm{Vc} \end{aligned}\right\} \rightarrow \text{Accumulation} = \frac{\Delta \mathrm{Vc}}{\Delta \mathrm{t}} \quad [\mathrm{MT}^{-1}]$$

If V is constant

$$\text{Accumulation} = \mathrm{V}\frac{\Delta \mathrm{c}}{\Delta \mathrm{t}} \rightarrow \mathrm{V}\frac{\mathrm{dc}}{\mathrm{dt}}$$

$$\frac{\mathrm{dc}}{\mathrm{dt}} > 0 \rightarrow \text{Increased accumulation}$$

$$\frac{\mathrm{dc}}{\mathrm{dt}} < 0 \rightarrow \text{Reduced accumulation}$$

$$\frac{\mathrm{dc}}{\mathrm{dt}} = 0 \rightarrow \text{Constant}$$

MASS BALANCE FOR A WELL-MIXED LAKE (cont)

❖Loading :

$$\text{Loading} = W(t) = Qc_{in}(t) \ \ [MT^{-1}]$$

Q : Flow$[L^3T^{-1}]$

$c_{in}(t)$: concentration$[ML^{-3}]$

❖ **Point source** (source identity, convenient measurement, continuality)

- + Municipal wastewater
- + Industrial wastewater
- + Tributary

❖ **Distribution source**

- Agriculture
- Atmosphere
- Water flowing from the city.
- Groundwater

Kết quả dịch

First order reaction coefficient

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MASS BALANCE FOR A WELL-MIXED LAKE (cont)

❖Outflow:

❖ **A WELL-MIXED LAKE** : $c = c_{out}$

$$\rightarrow \text{Outflow} = Qc_{out} = Qc$$

❖Reaction:

$$\text{Reaction} = kM = kVc$$

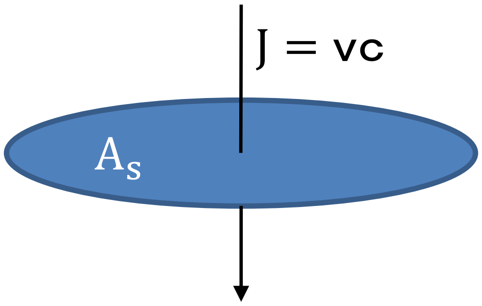
k: First order reaction coefficient [T^{-1}]

❖Settling:

$$\text{Settling} = vA_s c$$

$$V = A_s H$$

$$\rightarrow \text{Settling} = \frac{v}{H} Vc = k_s Vc$$



v: apparent settling velocity[LT^{-1}]

k_s: a first – oder settling rate constant= v/H [T^{-1}]

H: depth[L]

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MASS BALANCE FOR A WELL-MIXED LAKE (cont)

❖Mass balancing equation

$$V\frac{dc}{dt} = W(t) - Qc - kVc - vA_sc$$

- Dependent variable: c
- Independent variable : t
- Impact function: W(t), which represents external influence on the system
- Parameter: V, Q, k, v, và A_s

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MASS BALANCE FOR A WELL-MIXED LAKE (cont)

- **Mass balance equation**

$$V\frac{dc}{dt} = W(t) - Qc - kVc - vA_s c = 0$$

- **Solution** $\mathrm{c} = \frac{\mathrm{W}}{\mathrm{Q+kV+vA_s}}$ hoặc $\mathrm{c} = \frac{\mathrm{W}}{\mathrm{a}}$

- **Assimilation agent (self-cleansing ability)**

$$\mathrm{a} = \mathrm{Q} + \mathrm{kV} + \mathrm{vA_s}$$

STEADY - SYSTEM SOLUTION(cont)

Example 1:

A lake has the following characteristics: volume = 50,000 m^3, depth = 2m, inflow = outflow = 7500m^3/day, temperature = 25^0C. The lake receives the input of pollutants from three sources:

- a factory discharge of 50 kg/day
- a flux from the air with a load of 0.6 g/m^2/day
- the inflow stream has a concentration of 10mg/l

If the pollutant decay at the rate of 0.25/day at 20^0C (θ=1.05).

a) Calculation of assimilation agents.

b) Determine the steady – state concentration

c) Calculates the mass per time for each term in the mass balance and displays your results on a plot.

STEADY - SYSTEM SOLUTION(cont)

❖Transfer function

$$c = \frac{W}{Q + kV + vA_s} = \frac{Qc_{in}}{Q + kV + vA_s}$$

$$\rightarrow \frac{c}{c_{in}} = \beta = \frac{Q}{Q + kV + vA_s} : \text{Transfer function}$$

- Transfer function : it specified how the system input is transformed into an output
- $\beta \ll 1$: the mechanical cleaning of the lake will reduce the concentration of pollution (high assimilation capacity)
- $\beta \rightarrow 1$: weak lake cleaning mechanics

STEADY - SYSTEM SOLUTION(cont))

❖Residence time

❖ The residence time of a substance E represents the mean amount of time that a particle of E would reside in a system.

$$\tau_{\mathrm{E}} = \frac{\mathrm{E}}{|\mathrm{dE/dt}|_{\pm}}$$

E: Quantity of E in a specified volume [M or $\mathrm{ML^{-3}}$]

$|\mathrm{dE/dt}|_{\pm}$: absolute value of source or lake [$\mathrm{MT^{-1}}$or $\mathrm{ML^{-3}T^{-1}}$]

- The residence time of the water in the lake: $\tau = \frac{\mathrm{V}}{\mathrm{Q}}$
- The residence time of pollutants in the lake

$$\tau = \frac{\mathrm{Vc}}{\mathrm{Qc + kVc + v}A_s c} = \frac{V}{Q + kV + vA_s}$$

STEADY - SYSTEM SOLUTION(cont)

Example 2:

A lake has the following characteristics: volume = 50,000 m^3, depth = 2m, inflow = outflow = 7500m^3/day, temperature = 25^0C. The lake receives the input of pollutants from three sources:

- a factory discharge of 50 kg/day
- a flux from the air with a load of 0.6 g/m^2/day
- the inflow stream has a concentration of 10mg/l

If the pollutant decay at the rate of 0.25/day at 20^0C (θ=1.05).

a) Inflow concentration.

b) Transfer function

c) The residence time of the water

d) The residence time of pollutants

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STEADY - SYSTEM SOLUTION(cont)

Exercises: A lake has the following characteristics.:

Surface area = $2x10^5$ m^2, Medium depth = 3 m, Q in= Q out = 45000 m^3/day, inflow BOD concentration = 4mg/l, residence time = 2 week

A subdivision housing 1 000 people will discharges raw sewage into this system. Individual contributes about

150 x 3.785 litter/day

and 0.25 x 453.6 g/day.

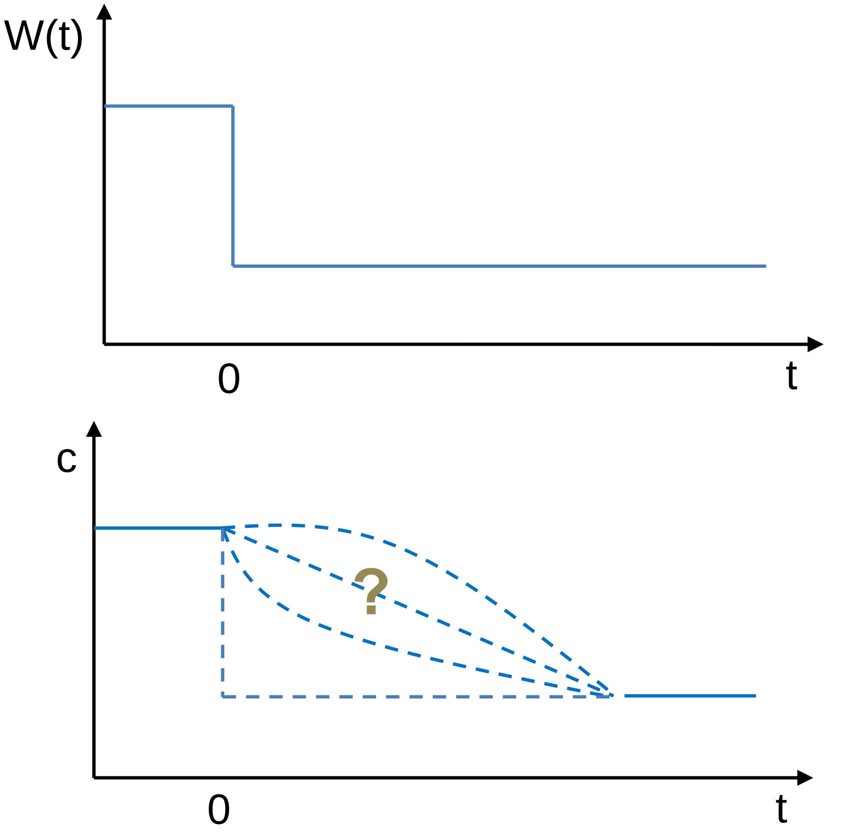
a) Determine the BOD concentration of wastewater (mg/l).

b) If BOD decay = 0.1 day^{-1} and settles at rate of 0.1 m/day. Calculate the assimilation factor for lake prior to building subdivision housing .

c) Calculation of transfer function factor after subdivision housing.

d) Determine the steady concentration of the lake in both cases with and without subdivision housing.

UNSTEADY- SYSTEMS SOLUTION (CONT)



4 water quality recovery scenarios with reduced pollutant load

At a certain point, if the reduction in pollutant emissions takes place, two questions will be asked as follows:

- How long will the water quality be restored?
- What is the "shape" of this recovery?

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UNSTEADY- SYSTEMS SOLUTION (CONT)

To determine the path of pollutants, consider the equation of mass balance:

$$\mathrm{V}\frac{\mathrm{dc}}{\mathrm{dt}} = \mathrm{W(t) - Qc - kVc - vA_s c}$$

$$\leftrightarrow \frac{\mathrm{dc}}{\mathrm{dt}} = \frac{\mathrm{W(t)}}{\mathrm{V}} - \frac{\mathrm{Q}}{\mathrm{V}}\mathrm{c} - \mathrm{kc} - \frac{\mathrm{v}}{\mathrm{H}}\mathrm{c}$$

$$\leftrightarrow \frac{\mathrm{dc}}{\mathrm{dt}} + \lambda\mathrm{c} = \frac{\mathrm{W(t)}}{\mathrm{V}} \qquad (*)$$

With $\lambda = \frac{Q}{V} + k + \frac{v}{H}$: eigenvalue

If Q, V, k, v, H is constant, equation (*) are differential equations, first order, linear, and heterogeneous. The solution consists of two parts:

$$\mathrm{c} = \mathrm{c_g} + \mathrm{c_p}$$

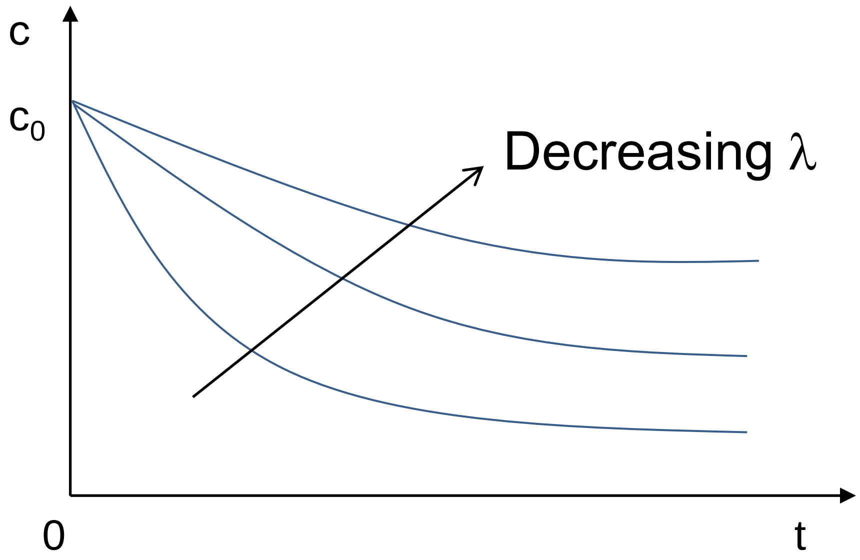
$\mathrm{c_g}$: Generalized solution for the case W(t) = 0

$\mathrm{c_p}$: particular solution for specified form W(t)

UNSTEADY- SYSTEMS SOLUTION (TT)

❖General slolution(c_g)

- W(t) = 0: $c = c_g$
- At t =0: $c = c_0$ → solution of equation (*) is: $c = c_0 e^{-\lambda t}$



The temporal response of well – mixed lake model following the termination of all loading

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UNSTEADY- SYSTEMS SOLUTION (CONT)

❖ Response time

- The time the lake needs to recover..
- t_ϕ = **Response time** ϕ%, which means the time it takes to achieve it. ϕ% the final level of recovery of the lake..

$$t_\phi = \frac{1}{\lambda} \ln \frac{100}{100 - \phi}$$

Response time	t_{50}	$T_{63.2}$	t_{75}	t_{90}	t_{95}	t_{99}
Formula	$0.693/\lambda$	$1/\lambda$	$1.39/\lambda$	$2.3/\lambda$	$3/\lambda$	$4.6/\lambda$

UNSTEADY- SYSTEMS SOLUTION (CONT)

Example 3

For a lake has the following characteristics**:** volume =50,000m^3, depth = 2m, Inflow = outflow = 7500m^3/day, temperature = 25^{0}C, waste loading= 140,000g/day, Decay rate= 0.319 d^{-1}

- If the initial concentration is equal to a steady - state level (5.97 mg/l), determine the general solution.
- Determine the response time 75%, 90%, 95%, and 99%

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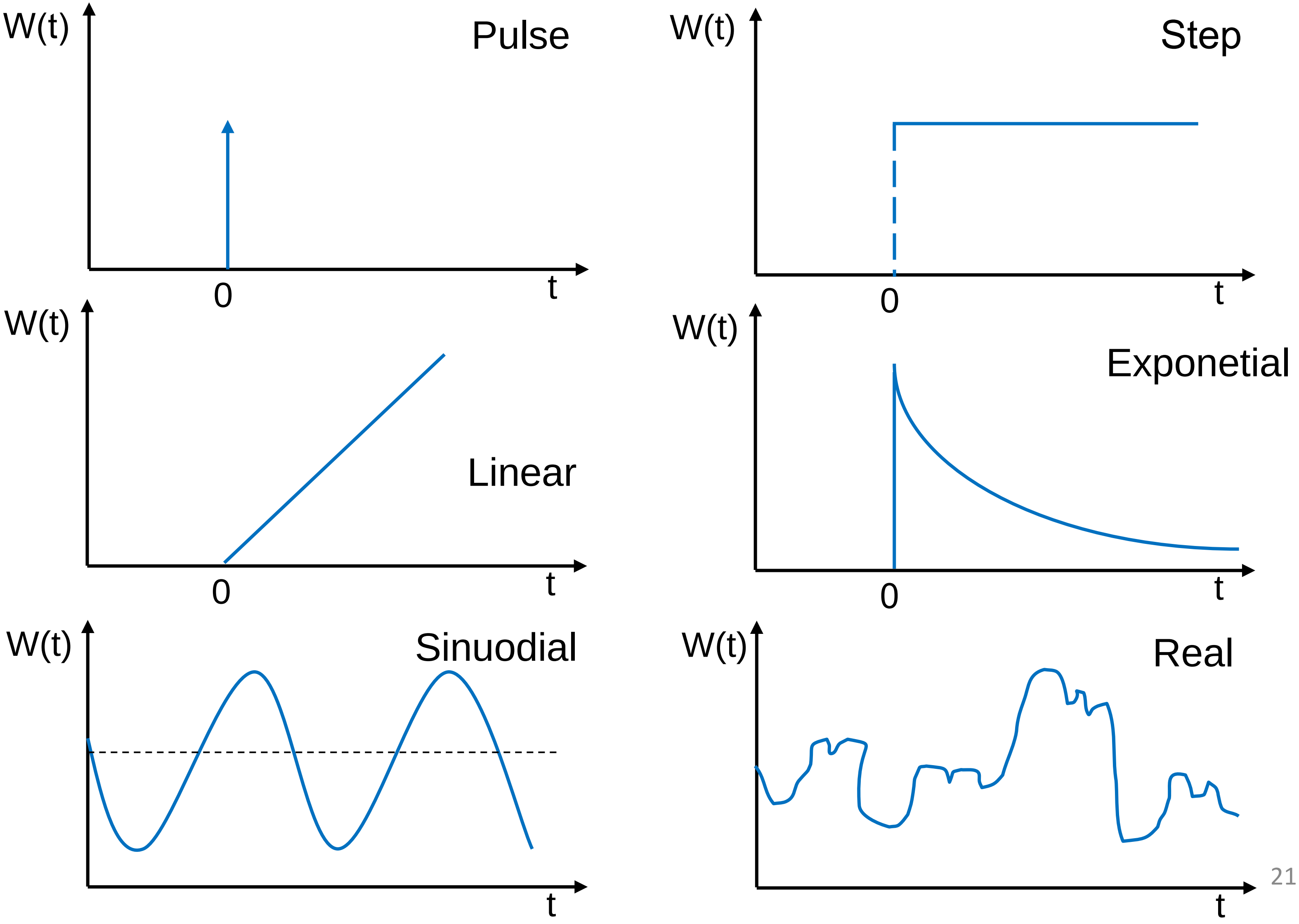
UNSTEADY- SYSTEMS SOLUTION (CONT)

- **Particular solutions(c_p): Loading function**



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UNSTEADY- SYSTEMS SOLUTION (CONT)

❖ **Particular solutions:** Integrating factor method.

$$\frac{dy}{dx} + p(y)x = g(y)$$

$$I = e^{\int p(y)dy}$$

$$\frac{dc}{dt} + \lambda c = \frac{W(t)}{V}$$

$I = e^{\int \lambda t} = e^{\lambda t}$

$$e^{\lambda t}\frac{dc}{dt} + e^{\lambda t}\lambda c = e^{\lambda t}\frac{W(t)}{V}$$

$\frac{d}{dt}\left(e^{\lambda t}c\right) = e^{\lambda t}\frac{dc}{dt} + e^{\lambda t}\lambda c$

$$\frac{dc}{dt}\left(e^{\lambda t}c\right) = e^{\lambda t}\frac{W(t)}{V}$$

$$e^{\lambda t}c\Big|_0^t = \frac{1}{V}\int e^{\lambda t}W(t)\,dt$$

$$ce^{\lambda t} - c_0 = \frac{1}{V}\int e^{\lambda t}W(t)\,dt$$

$$c(t) = \underbrace{\boxed{c_0 e^{-\lambda t}}}_{\mathbf{c_g}} + \underbrace{\boxed{\frac{e^{-\lambda t}}{V}\int e^{\lambda t}W(t)\,dt}}_{\mathbf{c_p}}$$

UNSTEADY- SYSTEMS SOLUTION (CONT)

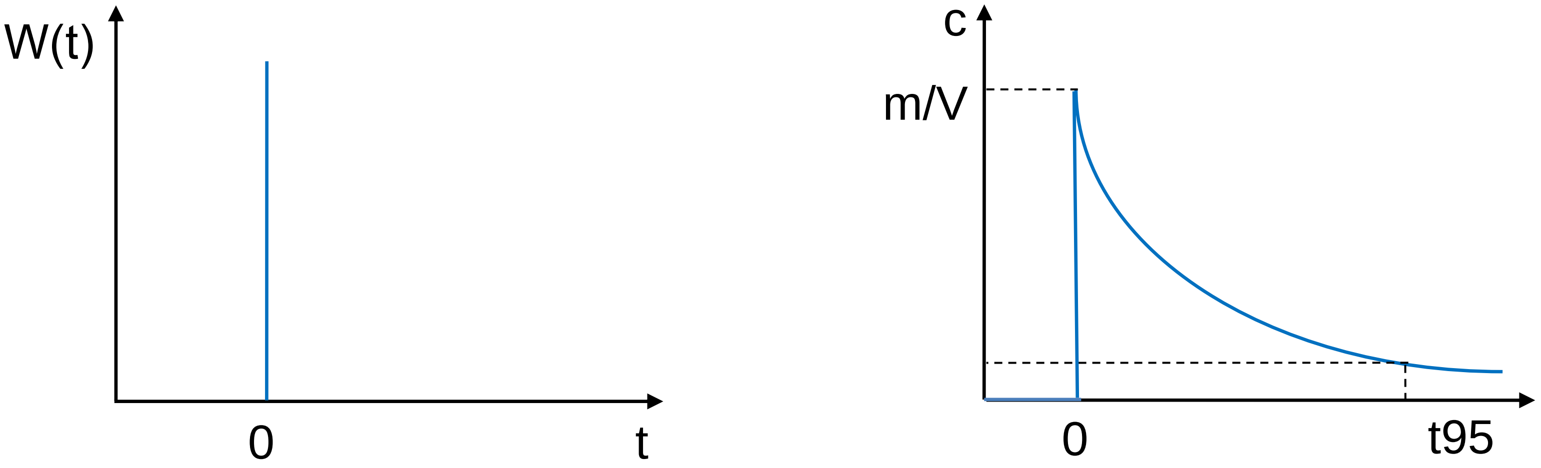
❖ IMPULSE LOADING (SPLII)

Describes the discharge that takes place in a short time.

Impulse function: $W(t) = m\delta(t)$ (m: quantity of pollutant mass[M])

Equation(*): $\frac{dc}{dt} + \lambda c = \frac{m\delta(t)}{V}$

Particular solution: $c = \frac{m}{V} e^{-\lambda t}$



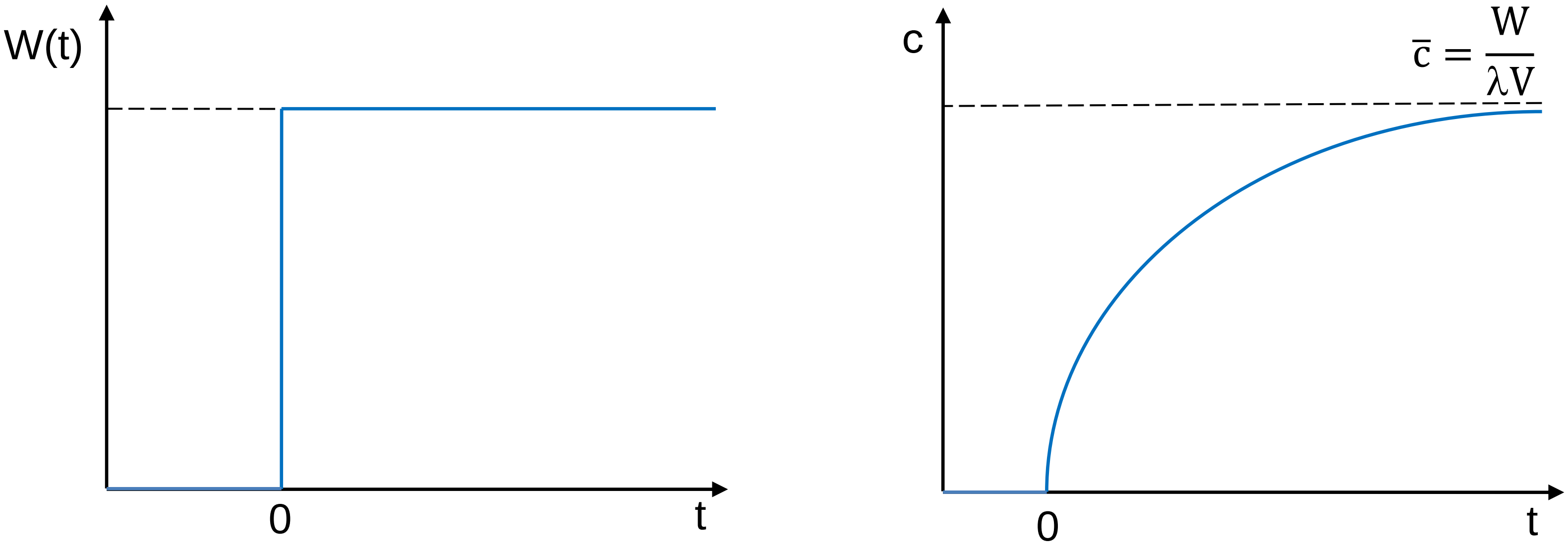
Plot of (a) loading and (b) response for impulse loading

UNSTEADY- SYSTEMS SOLUTION (CONT)

- **Step loading (new continuos source)**

- Step Funtion: $W(t) = \begin{cases} 0, & t < 0 \\ W, & t \geq 0 \end{cases}$ (W: Loading[MT^{-1}])

Particular solution : $c = \frac{W}{\lambda V}\left(1 - e^{-\lambda t}\right)$



Plot of (a) loading and (b) response for impulse loading

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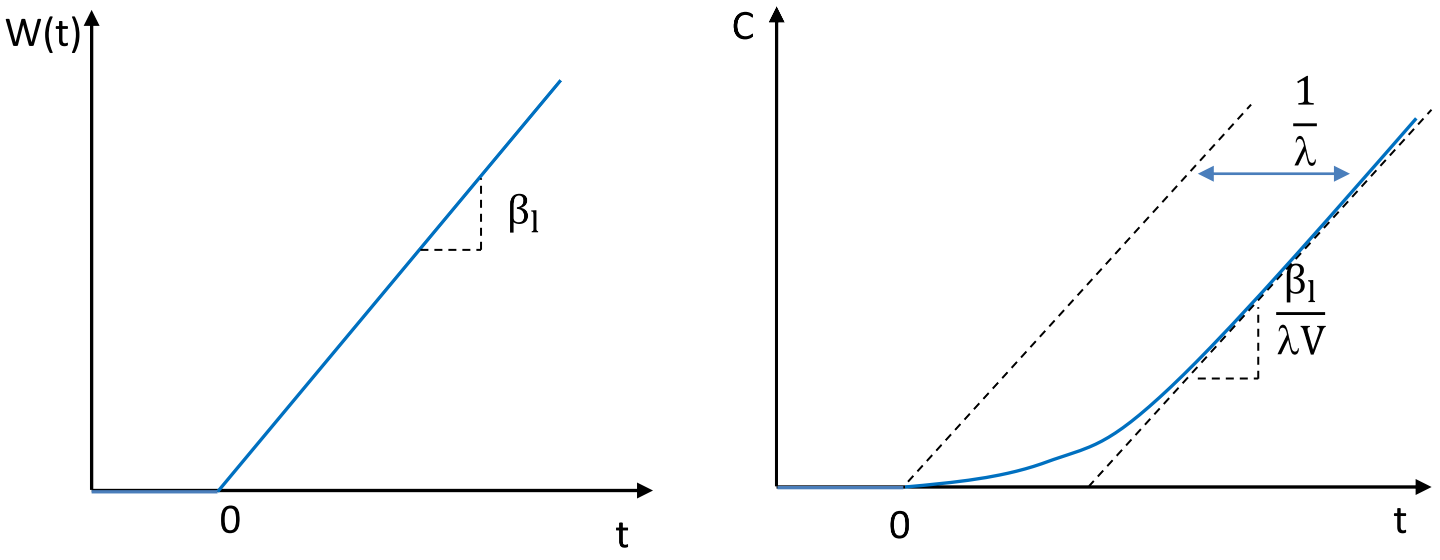
UNSTEADY- SYSTEMS SOLUTION (CONT)

- ❖ **Linear (Ramp) Loading**
- ❖ Linear function: $W(t) = \pm\beta_l t$ (β_l: rate of change[MT^{-2}])

Particular solution : $c = \pm\frac{\beta_l}{\lambda^2 V}\left(\lambda t - 1 + e^{-\lambda t}\right)$



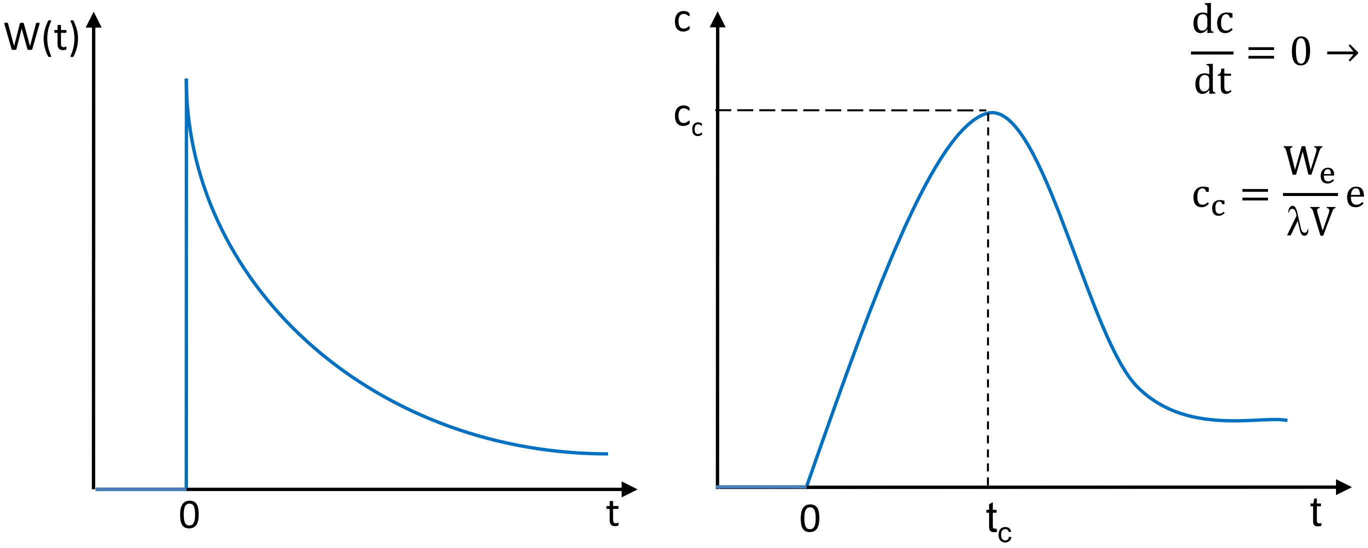
Plot of (a) loading and (b) response for a lineare increasing loading

UNSTEADY- SYSTEMS SOLUTION (CONT)

❖ Exponential loading

Exponential function: $W(t) = W_e e^{\pm\beta_e t}$ (W_e: a parameter that denote the value at t = 0 [MT^{-1}], β_e: growth rate (+) or decay (-) of loading [T^{-1}])

Particular solution : $c = \frac{W_e}{V(\lambda \pm \beta_e)}\left(e^{\pm\beta_e t} - e^{-\lambda t}\right)$



$$\frac{dc}{dt} = 0 \rightarrow t_c = \frac{\ln(\beta_e/\lambda)}{\beta_e - \lambda}$$

$$c_c = \frac{W_e}{\lambda V} e^{-\beta_e t_c} = \frac{W_e}{\lambda V}\left(\frac{\beta_e}{\lambda}\right)^{\frac{\beta_e}{\lambda - \beta_e}}$$

Plot of (a) loading and (b) response for an exponential decaying loading

EXERCISES

Exercise 1:

A lake receives two sources of waste of a preserved pollutant from two plants with the following characteristics:

Source of waste from factory 1 with the load of pollutants with the following characteristics:

$$\mathrm{W(t)} = \begin{cases} 0 & ,\mathrm{x} < 1930 \\ 13.2 \times 10^9(\mathrm{t} - 1930) & ,1930 \leq \mathrm{t} \leq 1960 \end{cases}$$

Waste source from factory 2 with pollutant load has the following characteristics:

$$\mathrm{W(t)} = \begin{cases} 0 & ,\mathrm{x} < 1900 \\ 229 \times 10^9 e^{0.015(t-1900)} & ,1900 \leq \mathrm{t} \leq 1960 \end{cases}$$

where W(t) has units g/year

The initial concentration of the lake is 3mg/l. Characteristics of the lake: outflow 49.1x10^9 m^3/year, V=4880x10^9 m^3. Determine the concentration of pollutants in the lake from 1900 to 1960.

Solution:

Because chloride is a conservative substance.:

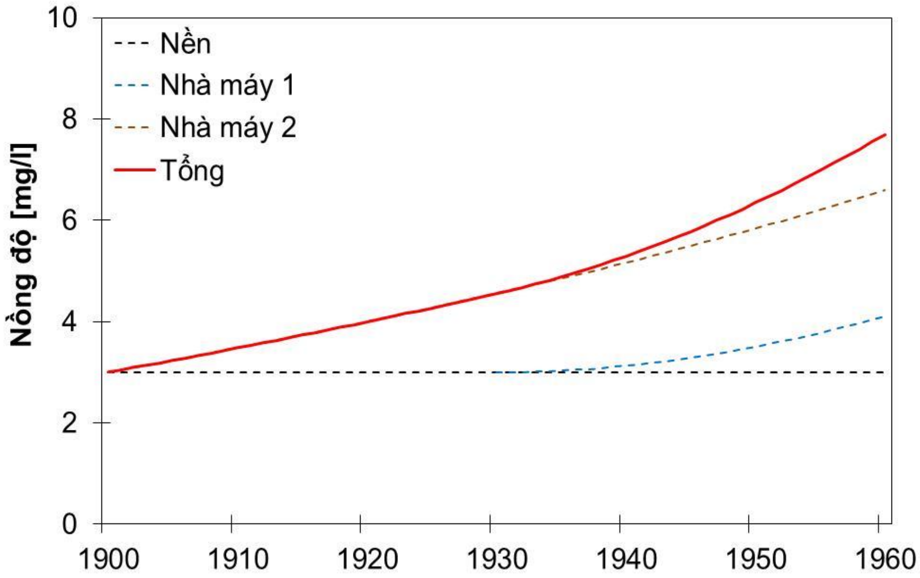
$$\lambda = \frac{Q}{V} = \frac{49.1 \times 10^9}{4880 \times 10^9} = 0.01\ (yr^{-1})$$

From 1900 to 1930

$$c = 3 + \frac{229 \times 10^9}{4880 \times 10^9 (0.01 + 0.015)} \left(e^{0.015(t-1900)} - e^{-0.01(t-1900)}\right)$$

From 1930 to 1960

$$\mathrm{c} = 3 + \frac{229 \times 10^9}{4880 \times 10^9 (0.01 + 0.015)} \left(e^{0.015(t-1900)} - e^{-0.01(t-1900)}\right) + \frac{13.2 \times 10^9}{0.01^2 \times 4880 \times 10^9} (-1 + e^{-0.01(t-1930)} + 0.01(t - 1930))$$



Concentration movement over time

EXERCISES

Exercise 2.

The lake has the following characteristics: Inflow = outflow = 5x10^5 m^3/year, volume = 4x10^7 m^3, surface area = 5x10^6 m^2. The initial concentration of the lake in a stable state is 5μg/l.

The lake receives two sources of waste as follows.:

- In 1994 the lake received a load of 500 kg/year from a fertilizer factory.
- In 1997, the lake began receiving waste from residential areas at a rate of population growth each year. $p = 200e^{0.2t}$ and each citizen will generate phosphorus amounts of 0.5 kg per year.

Calculate the concentration of the lake from 1994 to 2010. Know the deposition speed of phosphorus is 8m /year.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Lecture 7

PROCESS OF SUBSTANCE TRANSMISSION

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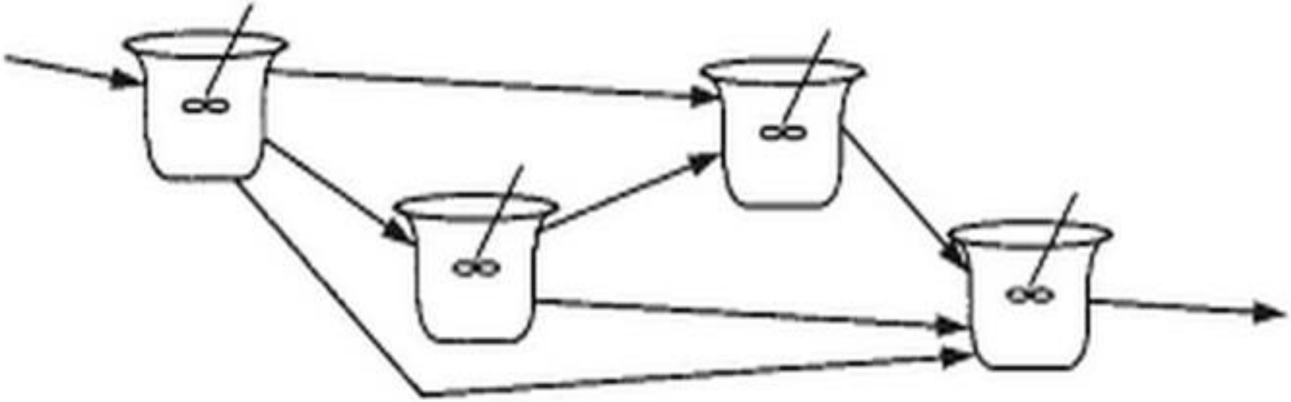
Lecture 7B

SYSTEM OF REACTOR

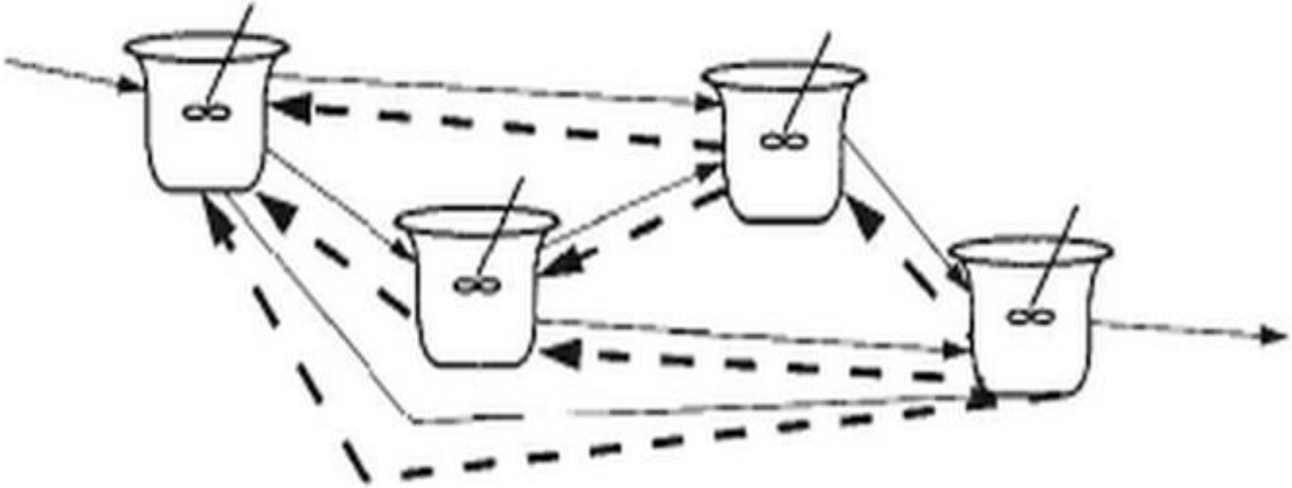
Lecturer: Professor. TS. Nguyen Ky Phung
Ms. Dang Thi Thanh Le

CONTENTS

- **Moving Impact System**
  - Mass Balance
  - Solution for the stable state
  - Solution for the unstable state
- **Reponses Impact System**
  - Mass Balance
  - Solution for the stable state
  - Solution for the unstable state



(a) **Moving Impact System**

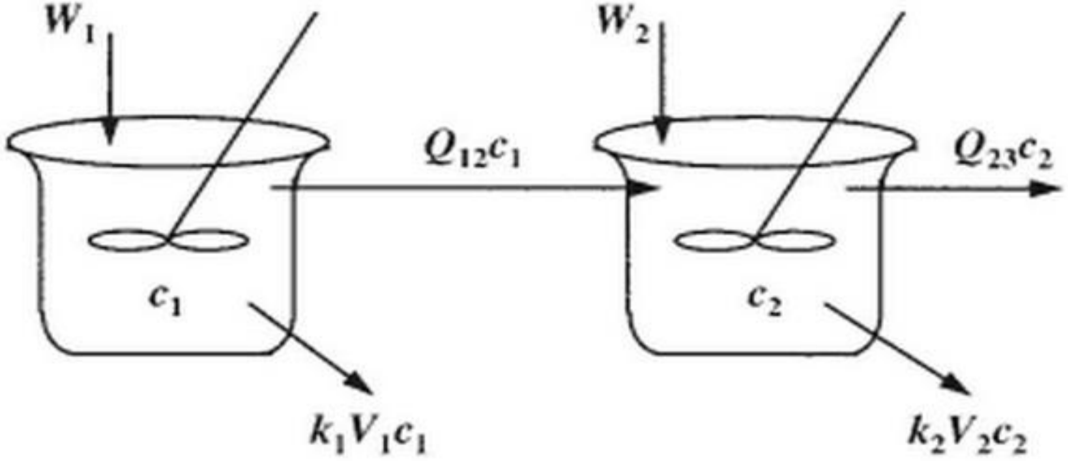


(b) **Reponses Impact System**

Moving Impact System

SOLUTION FOR THE STABLE STATE

A simple application is to simulate a chain of interconnected lakes using short river sections. In this case, we consider a simple system of two lakes as described:



Mass balance for these reactors can be written:

Lake 1 $$V_1\frac{dC_1}{dt}=W_1+Q_{01}C_0-Q_{12}C_1-k_1V_1C_1 \quad (1)$$

Lake 2 $$V_2\frac{dC_2}{dt}=W_2+Q_{12}C_1-Q_{23}C_2-k_2V_2C_2 \quad (2)$$

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SOLUTION FOR THE STABLE STATE

At steady – state:

$$a_{11}C_1 = W_1 \qquad (3)$$

$$-a_{21}C_1 + a_{22}C_2 = W_2 \qquad (4)$$

where

$$a_{11} = Q_{12} + k_1V_1 \qquad a_{21} = Q_{12} \qquad a_{22} = Q_{23} + k_2V_2$$

Solution

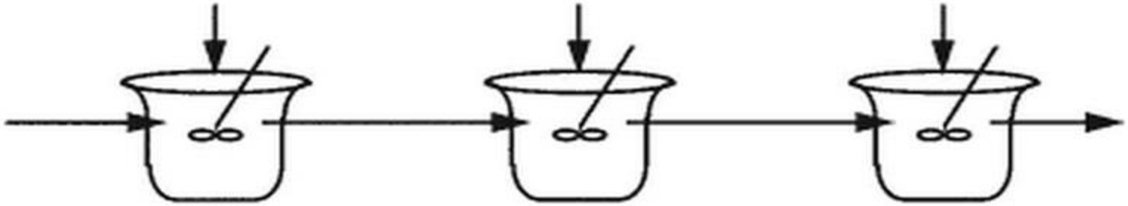
Lake 1 $$C_1 = \frac{1}{a_{11}}W_1 = \frac{1}{Q_{12} + k_1V_1}W_1 \qquad (5)$$

Lake 2 $$C_2 = \frac{W_2 + a_{21}C_1}{a_{22}} = \frac{1}{Q_{23} + k_2V_2}W_2 + \frac{Q_{12}}{Q_{23} + k_2V_2}\frac{1}{Q_{12} + k_1V_1}W_1 \qquad (6)$$

Example 1. Suppose that three lakes are connected in series. The pollutant settles at a rate of 10 m/yr.

(a) Determining concentrations in the steady- state of each reactor

(b) Determining how much the concentration in the third reactor to the loading of the second reactor.



		Lake 1		**Lake2**		**Lake 3**	
Volume, 10^6 m^3		2		4		3	
Depth, m		3		7		3	
Surface area, 10^6 m^2		0.667		0.571		1.000	
Loading, kg/yr		2000		4000		1000	
Flow, 10^6 m^3/yr	1.0		1.0		1.0		1.0

Solution

(a) The concentration of the lakes is determined as follows:

$$C_1 = \frac{W_1}{Q_{12} + vA_1} = \frac{2 \times 10^9}{1.0 \times 10^6 + \left(10 \times 0.667 \times 10^6\right)} = 260.76 \mu g L^{-1}$$

$$C_2 = \frac{W_2}{Q_{23} + vA_2} + \frac{Q_{12} C_1}{Q_{23} + vA_2} = \frac{4 \times 10^9}{1.0 \times 10^6 + \left(10 \times 0.571 \times 10^6\right)} + \frac{1.0 \times 10^6 \left(270.76\right)}{1.0 \times 10^6 + \left(10 \times 0.571 \times 10^6\right)}$$

$$= 596.13 + 38.86 = 634.99 \mu g L^{-1}$$

$$C_3 = \frac{W_3}{Q_{34} + vA_3} + \frac{Q_{23} C_2}{Q_{34} + vA_3} = \frac{1 \times 10^9}{1.0 \times 10^6 + \left(10 \times 1 \times 10^6\right)} + \frac{1.0 \times 10^6 \left(634.99\right)}{1.0 \times 10^6 + \left(10 \times 1 \times 10^6\right)}$$

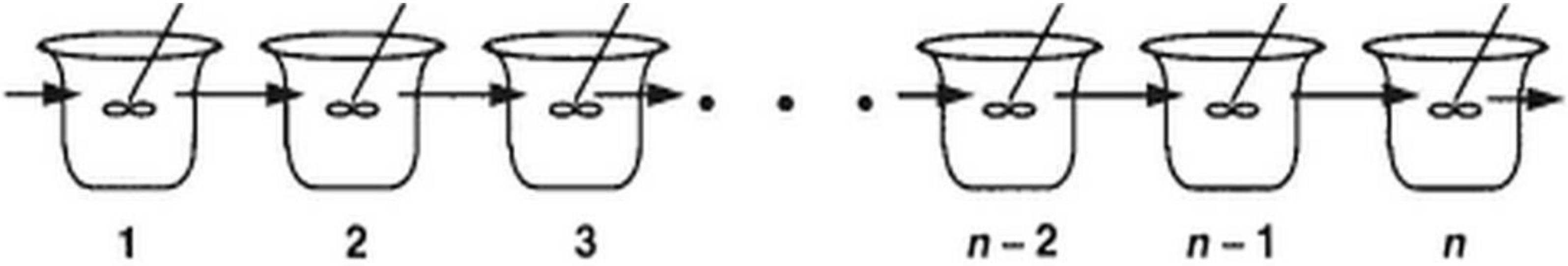
$$= 148.64 \mu g L^{-1}$$

(b) As the solution to C2 above we notice., 596.13μgL^{-1} of C_2 due to load into lake 2 (W_2), whereas 38.86μgL^{-1} it is due to loading into lake 1. Therefore, the effect of Lake 2 on Lake 3 can be calculated as follows:

$$\frac{1.0 \times 10^6 \left(596.13\right)}{1.0 \times 10^6 + \left(10 \times 1 \times 10^6\right)} = 54.19 \mu g L^{-1}$$

SOLUTION FOR THE STABLE STATE

A special case for the system of well-mixed lakes is when the volume and flow of the lakes are the same



Cascade model

The solution of the lakes is simplified as follows

Lake 1 $C_1 = \frac{Q}{Q+kV} C_0$

Lake 2 $C_2 = \frac{Q}{Q+kV} C_1 = \frac{Q}{Q+kV} \frac{Q}{Q+kV} C_0$

Lake n $C_n = \left(\frac{Q}{Q+kV} \right)^n C_0$

SOLUTION FOR THE UNSTABLE STATE

Equations (1) and (2) in case W = 0 are written as follows:

Lake 1 $\frac{dC_1}{dt} = -\lambda_{11} C_1$ (7)

Lake 2 $\frac{dC_2}{dt} = \lambda_{21} C_1 - \lambda_{22} C_2$ (8)

Where $\lambda_{11} = \frac{Q_{12}}{V_1} + k_1$ $\lambda_{21} = \frac{Q_{12}}{V_1}$ $\lambda_{22} = \frac{Q_{23}}{V_2} + k_2$

At t = 0: $C_1 = C_{10}$ and $C_2 = C_{20}$, We have the following solution.:

Lake 1 $C_1 = C_{10} e^{-\lambda_{11} t}$ (9)

Lake 2 $C_2 = C_{20} e^{-\lambda_{22} t} + \frac{\lambda_{21} C_{10}}{\lambda_{22} - \lambda_{11}} \left(e^{-\lambda_{11} t} - e^{-\lambda_{22} t} \right)$ (10)

SOLUTION FOR THE UNSTABLE STATE

To find a general solution to the system of lakes, Di Toro (1972) developed the following relationship:

Lake 1 $C_1(t,\lambda_{11}) = C_{10}e^{-\lambda_{11}t}$

Lake2 $C_2(t,\lambda_{11},\lambda_{22}) = \frac{\lambda_{21}}{\lambda_{22}-\lambda_{11}}\left[C_1(t,\lambda_{11}) - C_1(t,\lambda_{22})\right]$

Lake 3 $C_3(t,\lambda_{11},\lambda_{22},\lambda_{33}) = \frac{\lambda_{32}}{\lambda_{33}-\lambda_{22}}\left[C_2(t,\lambda_{11},\lambda_{22}) - C_1(t,\lambda_{11},\lambda_{33})\right]$

Lake 4 $C_4(t,\lambda_{11},\lambda_{22},\lambda_{33},\lambda_{44}) = \frac{\lambda_{43}}{\lambda_{44}-\lambda_{33}}\left[C_3(t,\lambda_{11},\lambda_{22},\lambda_{33}) - C_1(t,\lambda_{11},\lambda_{22},\lambda_{44})\right]$

Lake n $C_n(t,\lambda_{11},\ldots,\lambda_{n,n}) = \frac{\lambda_{n,n-1}}{\lambda_{n,n}-\lambda_{n-1,n-1}}\left[C_{n-1}(t,\lambda_{11},\ldots,\lambda_{n-2,n-2},\lambda_{n-1,n-1}) - C_{n-1}(t,\lambda_{11},\ldots.,\lambda_{n-2,n-2},\lambda_{n,n})\right]$

The general form of this factored version is

$$C_n(t,\lambda_{11},\ldots,\lambda_{n-1,n-1},\lambda_{n,n}) = \prod_{j=1}^{n-1}\lambda_{j+1,j}\sum_{i=1}^{n}\frac{C_1(t,\lambda_{i,i})}{\prod_{j=1(j\neq i)}^{n}(\lambda_{j,j}-\lambda_{i,i})}$$

Reponses Impact System

REPONSES IMPACT SYSTEM

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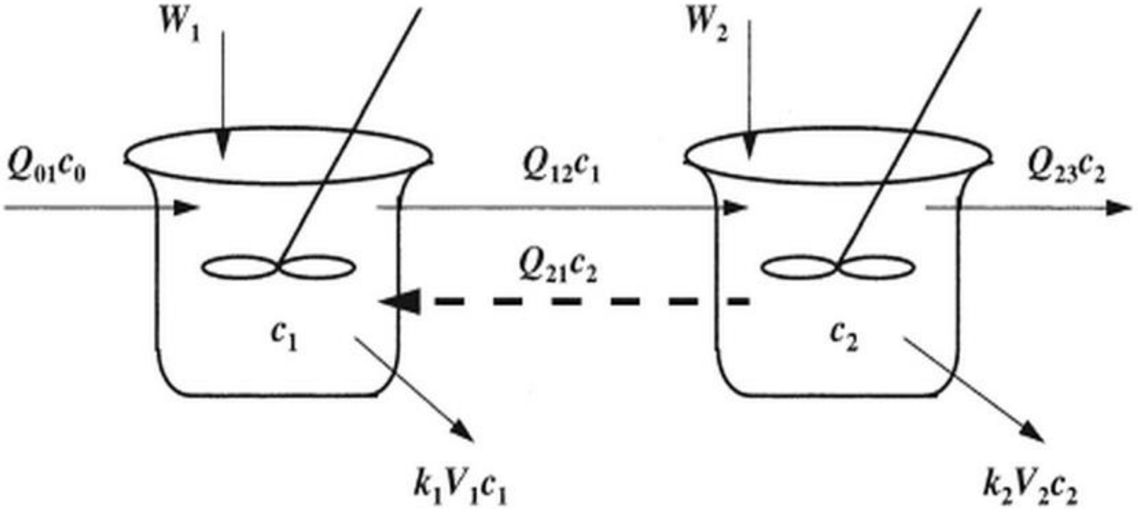
In this case, we consider a simple system of two lakes as described:



Mass balances for two CSTRs with feedback can be written

Lake 1 $$V_1\frac{dC_1}{dt}=W_1+Q_{01}C_0-Q_{12}C_1-k_1V_1C_1+Q_{21}C_2 \quad (11)$$

Lake 2 $$V_2\frac{dC_2}{dt}=W_2+Q_{12}C_1-Q_{21}C_2-Q_{23}C_2-k_2V_2C_2 \quad (12)$$

SOLUTION FOR THE STABLE STATE

At steady - state:

$$a_{11}C_1 + a_{12}C_2 = W_1 \quad (13)$$

$$a_{21}C_1 + a_{22}C_2 = W_2 \quad (14)$$

Where

$$a_{11} = Q_{12} + k_1V_1 \qquad a_{21} = -Q_{12}$$

$$a_{12} = -Q_{21} \qquad a_{22} = Q_{21} + Q_{23} + k_2V_2$$

Solution

Lake 1 $$C_1 = \frac{1}{a_{11} - (a_{21}a_{12}/a_{22})}W_1 + \frac{1}{a_{21} - (a_{11}a_{22}/a_{12})}W_2 \quad (15)$$

Lake 2 $$C_2 = \frac{1}{a_{12} - (a_{11}a_{22}/a_{21})}W_1 + \frac{1}{a_{22} - (a_{21}a_{12}/a_{11})}W_2 \quad (16)$$

SOLUTION FOR THE STABLE STATE

➢ The solution method for the system of reactor

Considering the response impact system of 3 lakes, the equation of the mass balance of system of reactor is written as follows:

$$a_{11}C_1 + a_{12}C_2 + a_{13}C_3 = W_1 \qquad (17)$$

$$a_{21}C_1 + a_{22}C_2 + a_{23}C_3 = W_2 \qquad (18)$$

$$a_{31}C_1 + a_{32}C_2 + a_{33}C_3 = W_3 \qquad (19)$$

The equation system is rewritten as follows: $[A]\{C\} = \{W\}$ (20)

where

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \{C\} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} \qquad \{W\} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \end{Bmatrix}$$

SOLUTION FOR THE STABLE STATE

➢ The solution method for the system of reactor (cont)

The solution of this equation system is as follows: $\{C\} = [A]^{-1}\{W\}$ (21)

where $[A]^{-1}$ is an inverse matrix. [A]

Using the two-matrix method, the equation (20) can be deployed as follows:

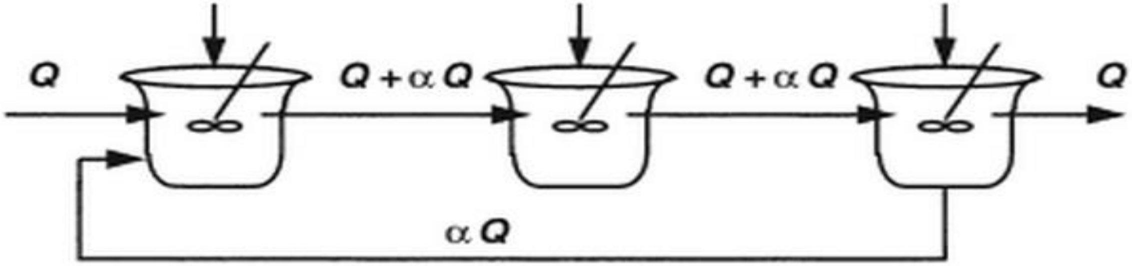
$$C_1 = a_{11}^{(-1)}W_1 + a_{12}^{(-1)}W_2 + a_{13}^{(-1)}W_3 \qquad (22)$$

$$C_2 = a_{21}^{(-1)}W_1 + a_{22}^{(-1)}W_2 + a_{23}^{(-1)}W_3 \qquad (23)$$

$$C_3 = a_{31}^{(-1)}W_1 + a_{32}^{(-1)}W_2 + a_{33}^{(-1)}W_3 \qquad (24)$$

Where $a_{i,j}^{(-1)}$ is coefficient in row i, column j in inverse matrix.

Example 2. Suppose that three lakes are connected in series



	Lake 1	**Lake 2**	**Lake3**
Volume, 10^6 m^3	2	4	3
Depth, m	3	7	3
Surface area, 10^6 m^2	0.667	0.571	1.000
Loading, kg/yr	2000	4000	1000

(a) If $Q = 1\times10^6$ m^3/yr, $\alpha = 0.5$ and the rate of settles of pollutants is 10 m/yr. Calculate the concentration of each lake..

(b) Use an inverse matrix to determine the concentration of Lake 3 due to the load to Lake 2 contributing.

(c) Identify inverse matrix in case where $\alpha = 0$

Lời giải

(a) The steady-state mass balances for the three reactors can be written as:

$$0 = W_1 - (Q + \alpha Q)C_1 - vA_1C_1 + \alpha QC_3$$

$$0 = W_2 + (Q + \alpha Q)C_1 - (Q + \alpha Q)C_2 - vA_2C_2$$

$$0 = W_3 + (Q + \alpha Q)C_2 - (Q + \alpha Q)C_3 - vA_3C_3$$

Substituting the parameter values, the three simultaneous equations can be expressed in matrix form as

$$\begin{bmatrix} 8.17\times10^6 & 0 & -0.5\times10^6 \\ -1.5\times10^6 & 7.21\times10^6 & 0 \\ 0 & -1.5\times10^6 & 11.5\times10^6 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 2\times10^9 \\ 4\times10^9 \\ 1\times10^9 \end{Bmatrix}$$

The inverse matrix is defined as follows

$$\begin{bmatrix} 1.23\times10^{-7} & 1.11\times10^{-9} & 5.33\times10^{-9} \\ 2.55\times10^{-8} & 1.39\times10^{-7} & 1.11\times10^{-9} \\ 3.33\times10^{-9} & 1.81\times10^{-8} & 8.71\times10^{-8} \end{bmatrix}$$

Which can be multiplied by (W) to give

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 255 \\ 608 \\ 166 \end{Bmatrix}$$

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(b) The effect of lake 2 on the concentration of lake 3 is determined based on the coefficient $a_{32}^{(-1)}$.

$$a_{32}^{(-1)} = 1.81 \times 10^{-8} \mu g.L^{-1} / mg.yr^{-1}$$

The concentration of lake 3 due to the loading of lake 2 contributes:

$(1.81 \times 10^{-8}) \times (4 \times 10^{9}) = 72.5\ \mu g/L$

(c) Where $\alpha = 0$, The equation system is written as a matrix as follows:

$$\begin{bmatrix} 7.67\times10^{6} & 0 & 0 \\ -1\times10^{6} & 6.71\times10^{6} & 0 \\ 0 & -1\times10^{6} & 11\times10^{6} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 2\times10^{9} \\ 4\times10^{9} \\ 1\times10^{9} \end{Bmatrix}$$

The inverse matrix is defined as follows:

$$\begin{bmatrix} 1.30\times10^{-7} & 0 & 0 \\ 1.94\times10^{-8} & 1.49\times10^{-7} & 0 \\ 1.77\times10^{-9} & 1.35\times10^{-8} & 9.09\times10^{-8} \end{bmatrix}$$

SOLUTION FOR THE UNSTABLE STATE

Equations (11) and (12) in case W = 0 are written as follows:

Lake 1 $$\frac{dC_1}{dt} = -\alpha_{11}C_1 + \alpha_{12}C_2 \qquad (25)$$

Lake 2 $$\frac{dC_2}{dt} = \alpha_{21}C_1 - \alpha_{22}C_2 \qquad (26)$$

Where

$$\alpha_{11} = \frac{Q_{12}}{V_1} + k_1 \qquad \alpha_{12} = \frac{Q_{21}}{V_1} \qquad \alpha_{21} = \frac{Q_{12}}{V_2} \qquad \alpha_{22} = \frac{Q_{23} + Q_{21}}{V_2} + k_2$$

At t = 0: $C_1 = C_{10}$ và $C_2 = C_{20}$, then the general solution can be developed:

Lake 1 $$C_1 = C_{1f}e^{-\lambda_f t} + C_{1s}e^{-\lambda_s t} \qquad (27)$$

Lake 2 $$C_2 = C_{2f}e^{-\lambda_f t} + C_{2s}e^{-\lambda_s t} \qquad (28)$$

SOLUTION FOR THE UNSTABLE STATE

Where

λ are eigenvalue and is defined as follows.

$$\begin{matrix}\lambda_f \\ \lambda_s\end{matrix} = \frac{(\alpha_{11}+\alpha_{22}) \pm \sqrt{(\alpha_{11}+\alpha_{22})^2 - 4(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})}}{2}$$

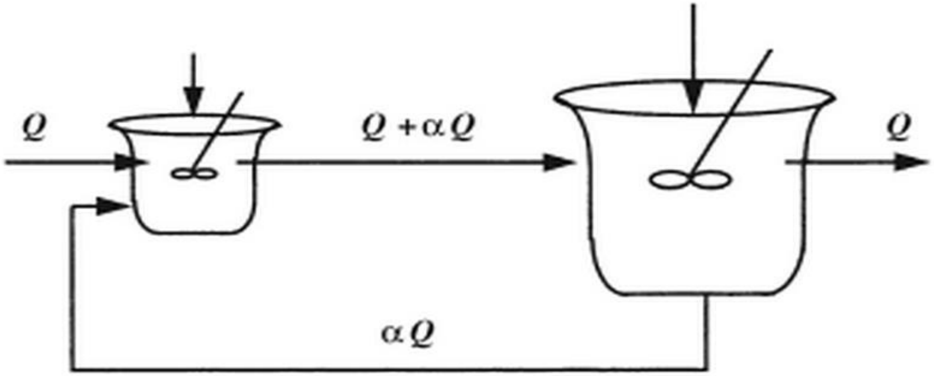
And the coefficients is

$$C_{1f} = \frac{(\lambda_f - \alpha_{22})C_{10} - \alpha_{12}C_{20}}{\lambda_f - \lambda_s} \qquad C_{1s} = \frac{\alpha_{12}C_{20} - (\lambda_s - \alpha_{22})C_{10}}{\lambda_f - \lambda_s}$$

$$C_{2f} = \frac{-\alpha_{21}C_{10} + (\lambda_f - \alpha_{11})C_{20}}{\lambda_f - \lambda_s} \qquad C_{2s} = \frac{-(\lambda_s - \alpha_{11})C_{20} + \alpha_{21}C_{10}}{\lambda_f - \lambda_s}$$

Based on the general solution, we realize that the resilience of the system depends on the magnitude of its own values. Note that $\lambda_f > \lambda_s$ và λ_f, λ_s Often used to refer to eigenvalues "fast" and "slow".

Example, 3. For a system of 2 lakes with the same characteristics as shown as below.



	Hồ 1	Hồ 2
Volume, 10^6 m^3	0.2	10
Depth, m	4	7
Surface area, 10^6 m^2	0.05	0.5
Load, kg/yr	2000	4000

(a) If $Q = 1 \times 10^6$ m^3/yr, $\alpha = 0.5$ and the rate of settles of pollutant is10 m/yr. Determined the concentration of each lake.

(b) Use the concentration value calculated in sentence (a) as the original value, determining the concentration movement over time of each lake.

Solution

(a) The solution is similar to example 2, the concentration in the steady- state of each lake:

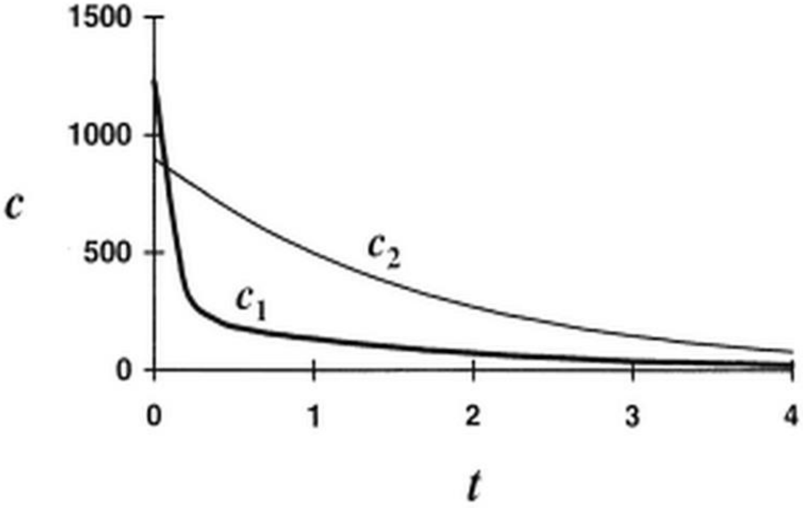
$$\{C\} = \begin{Bmatrix} 1224.5 \\ 898 \end{Bmatrix}$$

(b) The concentration of each lake over time is determined as follows

$$C_1 = 981.24e^{-10.04t} + 243.25e^{-0.61t}$$

$$C_2 = -15.67e^{-10.04t} + 913.63e^{-0.61t}$$

The chart shows the concentration of each lake as follows



Based on the graph, because of the large eigenvalue of Lake 1 at the beginning, the Lake 1 concentration rapidly decreased before slowly recovered

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LECTURE

MODELLING THE MARINE ENVIRONMENT

Lecturer: Prof. Nguyen Ky Phung

MSc. Dang Thi Thanh Le

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Lecture 7

THE SUBSTANCE TRANSMISSION PROCESS

Lecturer: Prof. Nguyen Ky Phung
MSc. Dang Thi Thanh Le

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Lecture 7C

CALCULATION METHODOLOGY PROFICIENT FUSION SYSTEMS

Lecturer: Prof. Nguyen Ky Phung
MSc. Dang Thi Thanh Le

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CONTENT

- Euler Method
- Heun Method
- Runge-Kutt Method

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Limitations of the analytical method

- *Non Idealized loading functions: Usually using the ideal load functions to achieve a closed solution. Although pollutant loads rarely acquire those functions, most are in any form and unpattern*
- *Variable parameter: In the previous chapters, there is a hypothesis that model parameters such as Q, V, k, etc. are constant. The reality shows that these parameters often change over time.*
- *Multiple–Lake systems: The system that consists of more than 2 lakes uses computers to accomplish an effective solution.*
- *Non-linear kinetics: Despite the vital of first-order kinetics, some water quality problems will require these kinetic reactions to be in a nonlinear state, or the analytical solutions are not acquired.*

Euler Method

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The complete mixed lake model

$$\frac{dC}{dt}=\frac{W(t)}{V}-\lambda C \qquad (1)$$

where

$$\lambda=\frac{Q}{V}+k+\frac{v}{H} \qquad (2)$$

Use the forward difference, we can approximate the first derivative of c with respect to t by :

$$\frac{dC_i}{dt}\cong\frac{\Delta C}{\Delta t}=\frac{C_{i+1}-C_i}{t_{i+1}-t_i} \qquad (3)$$

Substituting (3) into the equation (1):

$$\frac{C_{i+1}-C_i}{t_{i+1}-t_i}=\frac{W(t)}{V}-\lambda C_i \qquad (4)$$

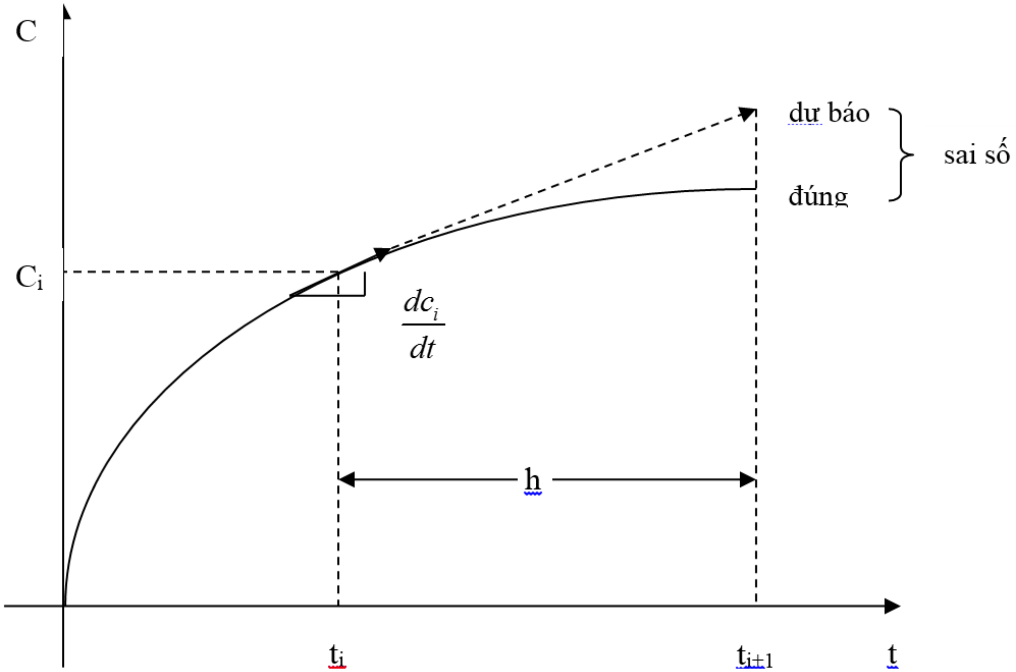
Euler Method

$$C_{i+1} = C_i + \left[\frac{W(t)}{V} - \lambda C_i\right](t_{i+1} - t_i) \qquad (5)$$

This method can be generalized as follows:

$$C_{i+1} = C_i + f(t_i, C_i)h \qquad (6)$$

Where $f(t_i, C_i) = dC_i/dt$ Tthe value calculated at t_i và C_i, và $h = t_{i+1\,–}\,t_{i.}$



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Example 1. A well-mixed lake has the following characteristics: Q = 10^5 m^3/yr, V = 10^6 m^3, z = 5 m, k = $0.2 yr^{-1}$, v = 0.25 m/yr.

At t = 0, the lake receives a loading of 50x10^6 g/yr and the lake has an initial concentration is 15 mg/L. Use the Euler method to simulate the concentration from the moment t = 0 to 20 year. With a time step of 1 year, Compare the results with the analytical solution:

$$C = C_0 e^{-\lambda t} + \frac{W}{\lambda V}\left(1 - e^{-\lambda t}\right)$$

Solution

(a) Eigenvalue is calculated as follows

$$\lambda = \frac{10^5}{10^6} + 0.2 + \frac{0.25}{5}(yr^{-1})$$

At the beginning (t =0), the concentration of the lake is 15 mg/L and receive a loading is 50x10^6 g/yr.

At t=1 year

$$C(1) = 15 + \left[\frac{50 \times 10^6}{10^6} - 0.35(15) \right] \times 1.0 = 59.75(mg/l)$$

At t=2 year

$$C(2) = 59.75 + \left[\frac{50 \times 10^6}{10^6} - 0.35(59.75) \right] \times 1.0 = 88.8375(mg/l)$$

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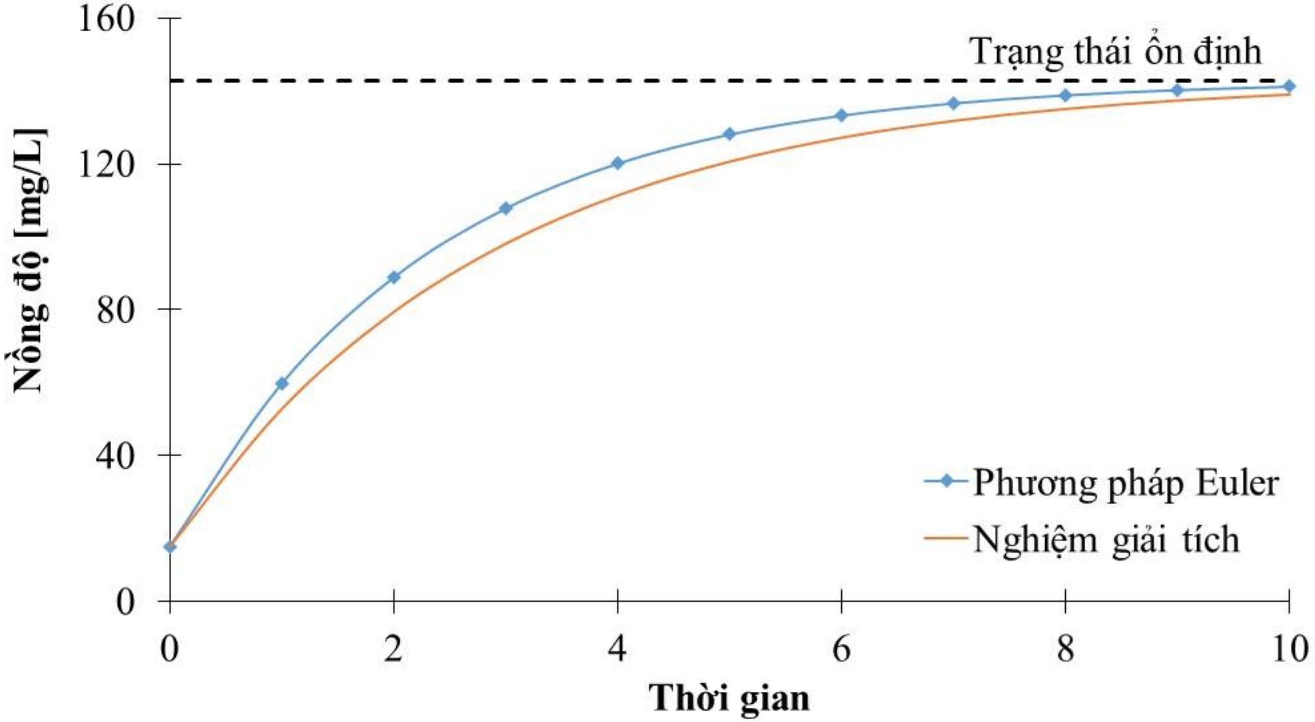
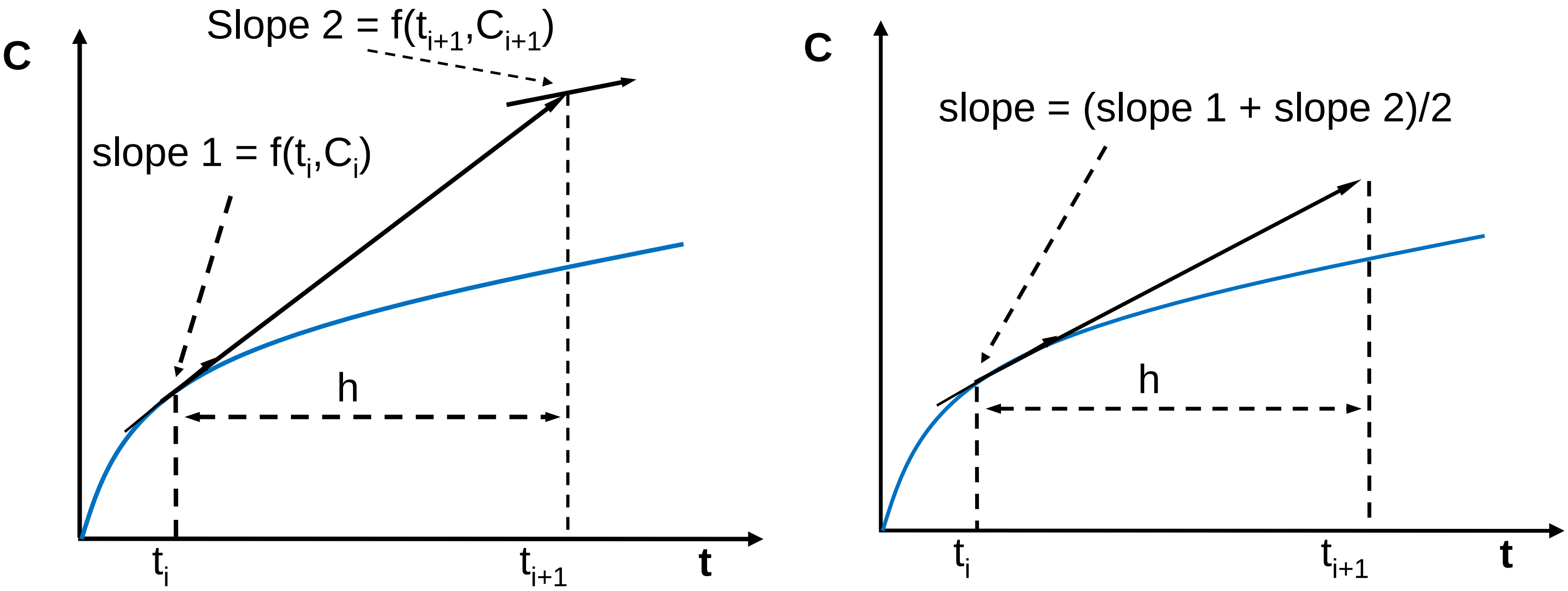


Graph comparing the euler method solution and the analytical solution

t (yr)	C (mg/L)		t (yr)	C (mg/L)	
	PP Euler	Analytical method		PP Euler	Analytical method
0	15.00	15.00	6	133.22	127.20
1	59.75	52.75	7	136.59	131.82
2	88.84	79.37	8	138.78	135.08
3	107.75	98.12	9	140.21	137.38
4	120.04	111.33	10	141.14	139.00
5	128.03	120.64	∞	142.86	142.86

HEUN METHOD

One method to improve the error in the Euler method is to identify the derivative on either side of a time step - one at the beginning (t_i) and one at the end (t_{i+1}). These two derivative components are then taken on average to estimate the slope of the entire time step (h)



(a) forecast

(b) calibration

Graphical describing the Heun method

HEUN METHOD

The Heun method is also known as the forecasting-calibratation method. This method is summarized as follows:

Forecast:

$$C_{i+1}^{0} = C_i + f\left(t_i, C_i\right)h \qquad (7)$$

Calibratation *h:*

$$C_{i+1} = C_i + \frac{f\left(t_i, C_i\right) + f\left(t_{i+1}, C_{i+1}^{0}\right)}{2}h \qquad (8)$$

Example 2. A well-mixed lake has the following characteristics: $Q = 10^5$ m^3/yr, $V = 10^6$ m^3, $z = 5$ m, $k = 0.2yr^{-1}$, $v = 0.25$ m/yr.

At t = 0, the lake receives a loading of $50x10^6$ g/yr and the lake has an initial concentration is 15 mg/L. Use the Huen method to simulate the concentration from the moment t = 0 to 20 year With a time step of 1 year. Compare the results with the analytical solution :

$$C = C_0 e^{-\lambda t} + \frac{W}{\lambda V}\left(1 - e^{-\lambda t}\right)$$

Solution

(a) Eigenvalue is calculated as follows

$$\lambda = \frac{10^5}{10^6} + 0.2 + \frac{0.25}{5}(yr^{-1})$$

At the beginning (t =0) The concentration of the lake is 15 mg/L and receive a loading 50x10^6 g/yr.

At t=1 year

$$C^0(1) = 15 + \left(50 - 0.35 \times 15\right) \times 1 = 59.75(mg/l)$$

$$C(1) = 15 + \frac{\left(50 - 0.35 \times 15\right) + \left(50 - 0.35 \times 59.75\right)}{2} \times 1 = 51.92(mg/l)$$

The same solution for other times

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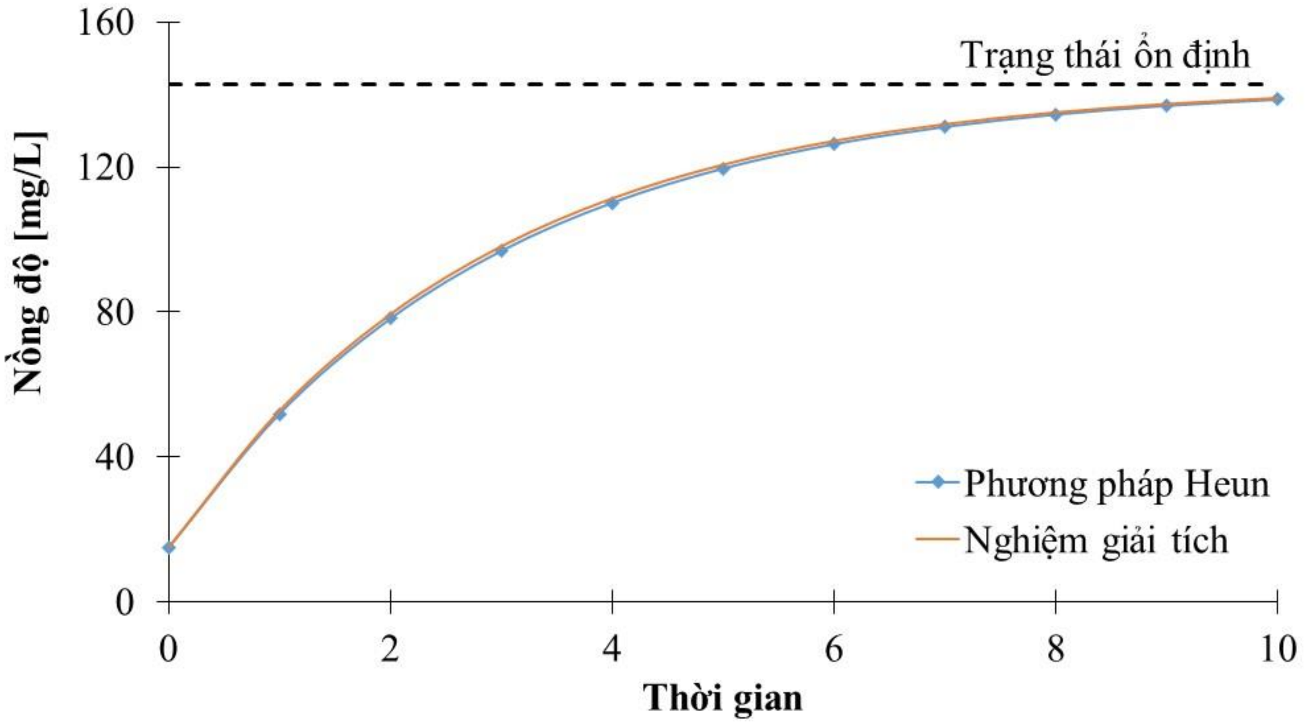


Graph comparing the solution by the Heun method and the Analytical solution

t (yr)	C (mg/L)		t (yr)	C (mg/L)	
	PP Heun	Analytical method		PP Heun	Analytical method
0	15.00	15.00	6	126.31	127.20
1	51.92	52.75	7	131.09	131.82
2	78.18	79.37	8	134.49	135.08
3	96.86	98.12	9	136.91	137.38
4	110.14	111.33	10	138.63	139.00
5	119.59	120.64	∞	142.86	142.86

Runge-Kutta Method

The Runge-Kutta method is generalized.: $$C_{i+1} = C_i + \phi h \quad (9)$$

ϕ is the slope (commonly referred to as the home function)

The most commonly used Runga-Kutta method is the 4th order method and has the following form:

$$C_{i+1} = C_i + \left[\frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)\right]h \quad (10)$$

Where

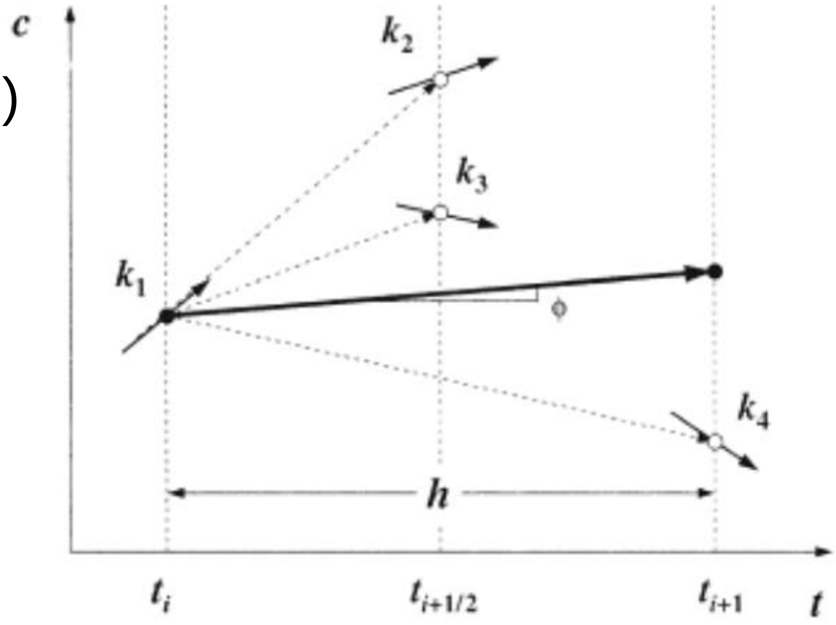
$$k_1 = f\left(t_i, C_i\right)$$

$$k_2 = f\left(t_i + \frac{1}{2}h, C_i + \frac{1}{2}hk_1\right)$$

$$k_3 = f\left(t_i + \frac{1}{2}h, C_i + \frac{1}{2}hk_2\right)$$

$$k_4 = f\left(t_i + h, C_i + hk_3\right)$$

$$f\left(t, C\right) = \frac{dC}{dt}\left(t, C\right)$$



Graph describing the Rungge-Kutta method

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Runge-Kutta Method

Butcher et al. (1964) developed the 5step Runge-Kutta method as follows:

$$C_{i+1} = C_i + \left[\frac{1}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right)\right]h \qquad (11)$$

where

$$k_1 = f\left(t_i, C_i\right)$$

$$k_2 = f\left(t_i + \frac{1}{4}h, C_i + \frac{1}{4}hk_1\right)$$

$$k_3 = f\left(t_i + \frac{1}{4}h, C_i + \frac{1}{8}hk_1 + \frac{1}{8}hk_2\right)$$

$$k_4 = f\left(t_i + \frac{1}{2}h, C_i - \frac{1}{2}hk_2 + hk_3\right)$$

$$k_5 = f\left(t_i + \frac{3}{4}h, C_i + \frac{3}{16}hk_1 + \frac{9}{16}hk_4\right)$$

$$k_6 = f\left(t_i + h, C_i - \frac{3}{7}hk_1 + \frac{2}{7}hk_2 + \frac{12}{7}hk_3 - \frac{12}{7}hk_4 + \frac{8}{7}hk_5\right)$$

Example 3. A well-mixed lake has the following characteristics: Q = 10^5 m^3/yr, V = 10^6 m^3, z = 5 m, k = 0.2yr^{-1}, v = 0.25 m/yr.

At t = 0, the lake receives a loading of 50x10^6 g/yr and the lake has an initial concentration is 15 mg/L. Use the Runnge method to simulate the concentration from the moment t = 0 to 20 year With a time step of 1 year. Compare the results with the Analytical method :

$$C = C_0 e^{-\lambda t} + \frac{W}{\lambda V}\left(1 - e^{-\lambda t}\right)$$

System equations

The above methods described can easily be adapted to simulate a system of difference equations of the form of:

$$\frac{dC_1}{dt} = f_1(C_1, C_2, \ldots, C_n) \qquad (12)$$

$$\frac{dC_2}{dt} = f_2(C_1, C_2, \ldots, C_n) \qquad (13)$$

$$\frac{dC_n}{dt} = f_n(C_1, C_2, \ldots, C_n) \qquad (14)$$

To solve this equation system, there is requires “n” as the initial condition at the time of starting the calculation

Exercise 1. A 5kg spill of dissolved pesticides occurred in Lake 1 in a system of two lakes. Know that both lakes are perfectly mixed. The characteristics of the lake are as follows:

	Lake 1	Lake 2
Volume (m^3)	0.5×10^6	0.6×10^6
Output (m^3/yr)	1×10^6	1×10^6

The concentration over time of the two lakes is forecast using the Euler method. Compare the results with the analytical solution and present the results by the graph.

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LECTURE

MODELLING THE MARINE ENVIRONMENT

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Lecture 7

PROCESS OF SUBSTANCE TRANSMISSION

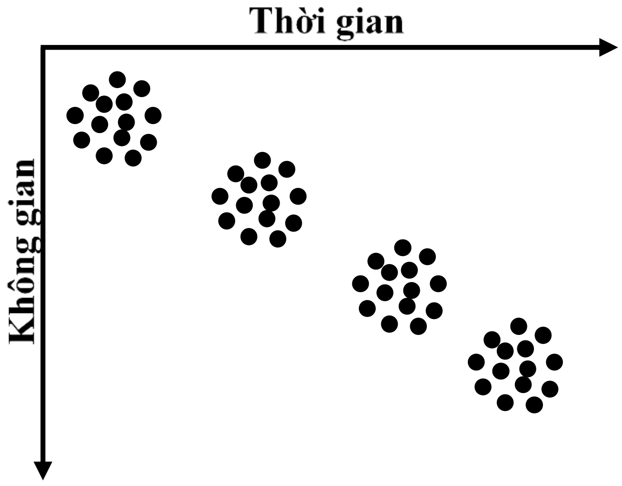
Lecturer: Prof. Nguyen Ky Phung
MSc. Dang Thi Thanh Le

Lecture 7D

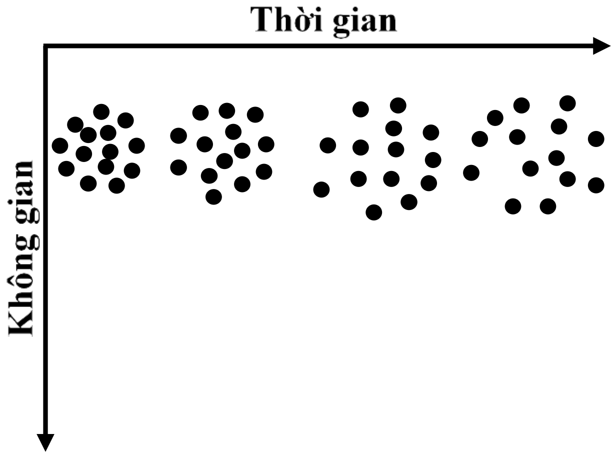
ADVECTION AND DISUSION

ADVECTION AND DISUSION

- ❖ Advection: Advection results from flow that is unidirectional and does not change the identity of the substance being transported. As in Figure advection moves matter from one position in space to another
- ❖ Diffusion: Refer to the moment of the mass due to random water motion or mixing
  - **Molecular diffusion**
  - **Turbulent diffusion**



(a) Advection



(b) Diffustion

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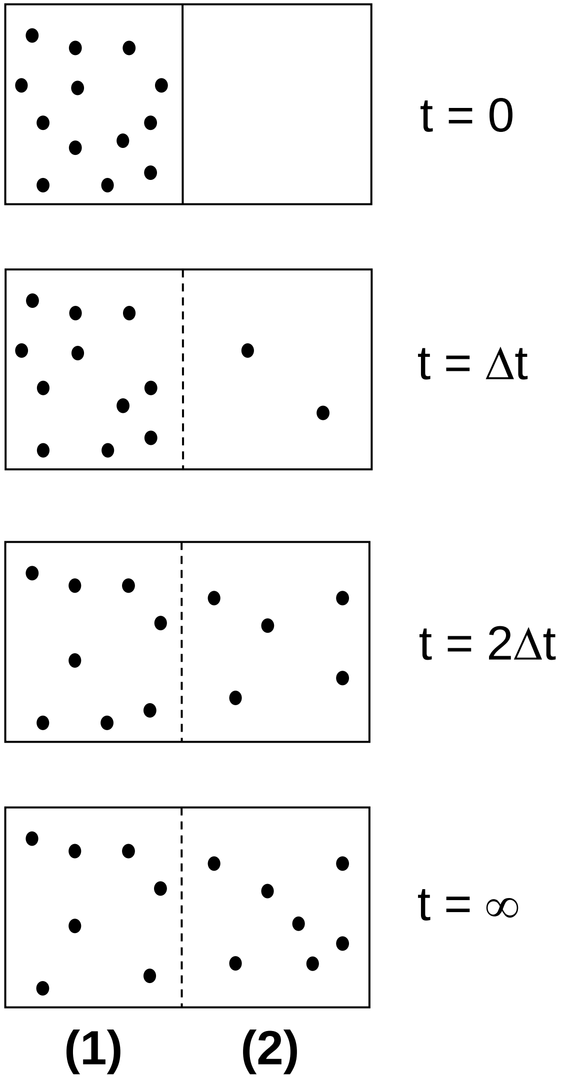
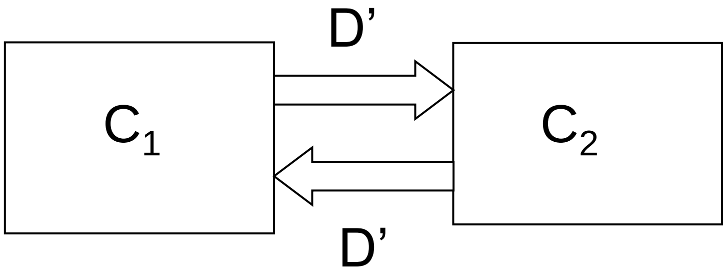
DIFFUSION

- Experiment





Mass equation balance for lake1:

$$V_1 \frac{dC_1}{dt} = D'(C_2 - C_1)$$

Where:

V_1 : Lake volume 1;

C_1, C_2 : concentration of 1 and 2;

D’ : Diffusion [m^3/năm]

DIFFUSION

❖ **Three factors affect diffusion:**

- The mixing flow D’ reflects the intensity of the mixing. Thus, if the tanks was subjects to only weak mixing such as due to Brownian motion, D’ would be small. If it were subjected to vigorous phýical mixing, D’ would be large
- Mass transport is directly proportional to the interface area
- Mass transport is proportional to the difference in concentration between the two lakes (gradient concentration)
- Diffuse to the right if $C_1 > C_2$
  + Diffuse to the left if C1 < C2
  + Do not diffuse if C1 = C

<u>Example 1.</u> Simulation of the time required for the experiment described on Figure 2 to be completed 95%.

Lời giải:

Mass balance equation

$$\frac{V}{2}\frac{dC_1}{dt} = D'\left(C_2 - C_1\right)$$

$V_1 = V_{2,}$: $C_{10} = C_1 + C_2$

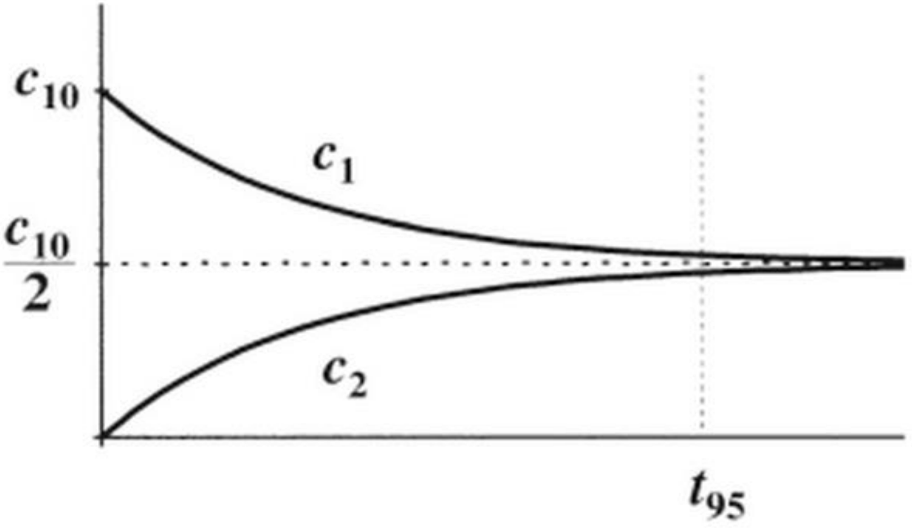
Mass balancing equations at the moment:

$$\frac{dC_1}{dt} + \frac{4D'}{V}C_1 = \frac{2D'}{V}C_{10}$$

Which can be solved

$$C_1 = C_{10}e^{-\frac{4D'}{V}t} + \frac{C_{10}}{2}\left(1 - e^{-\frac{4D'}{V}t}\right)$$

$$C_2 = \frac{C_{10}}{2}\left(1 - e^{-\frac{4D'}{V}t}\right)$$



$$t_{95} = \frac{3}{\lambda} = \frac{3V}{4D'}$$

FICK’S FIRST LAW

In 1855, physicist Adolf Fick introduced the diffusion model:

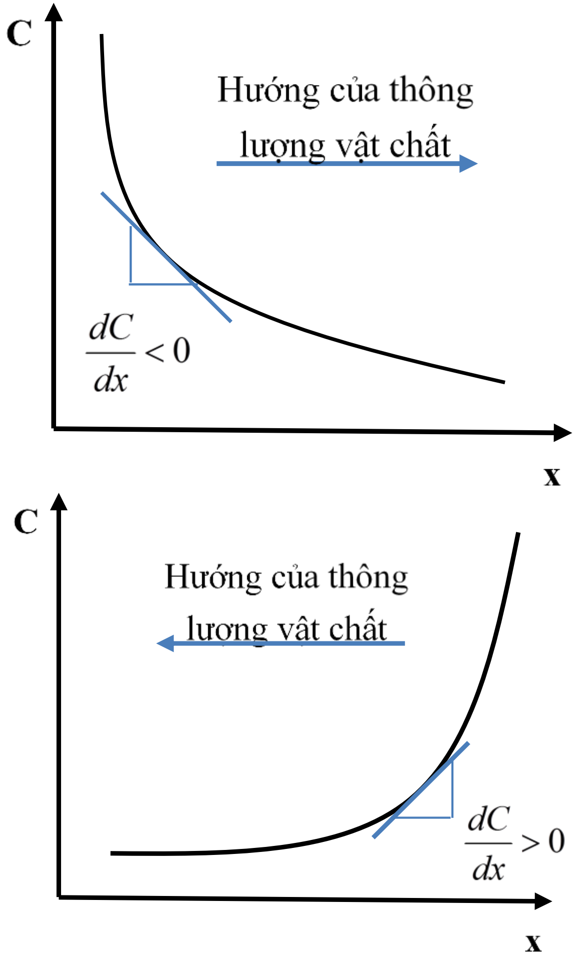
$$J_x = -D\frac{dC}{dx}$$

Where:

J_x : mass flux in the x ditection[$ML^{-2}T^{-1}$]

D: diffusion coefficient [L^2T^{-1}]

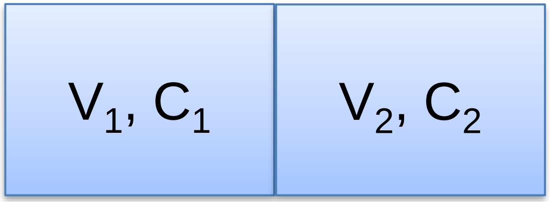
Fick's Law: Mass flux is proportional to the gradient (that is, the derivative or rsate of change)



Graphical depictions the effect of concentration gradients on the mass flux.

FICK’S FIRST LAW

❖ Determination of diffusion coefficient



$$V_1 \frac{dc_1}{dt} = -JA_c$$

l: mixing length [L]

$$J_x = -D\frac{dc}{dx}$$

$$\frac{dc}{dx} \cong \frac{c_2 - c_1}{l}$$

Combining these equations

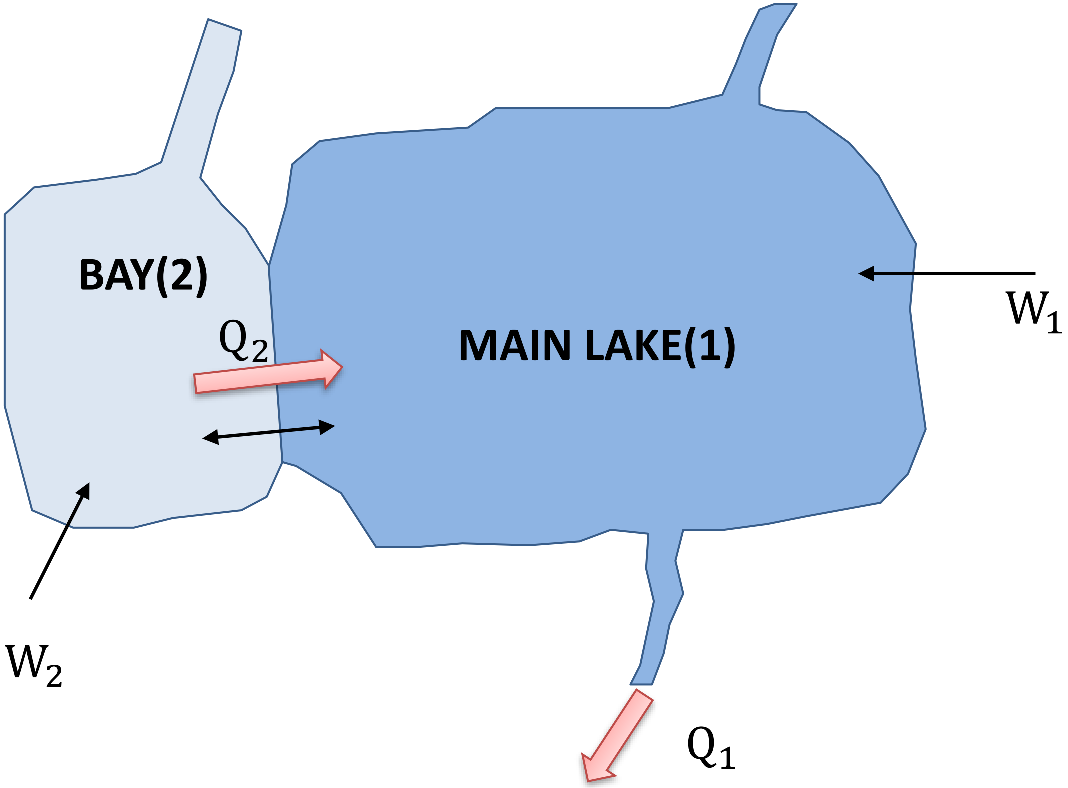
$$V_1 \frac{dc_1}{dt} = \underbrace{\boxed{\frac{DA_c}{l}}}_{D'}(c_2 - c_1)$$

In diffusion, two important quantities are d diffusion coefficient and blend length l. For the molecular, the parameter is defined [LT^{-1}]:

$v_d = \frac{D}{l}$ where v_d is called a diffusion mass – transfer coefficient

For, **Turbulent diffusion** : $E' = \frac{EA_c}{l}$

EMBAYMENT MODEL



Mass balancing equation:

Main lake

$$V_1 \frac{dC_1}{dt} = W_1 - Q_1C_1 - k_1V_1C_1 + Q_2C_2 + E'\left(C_2 - C_1\right)$$

Bay

$$V_2 \frac{dC_2}{dt} = W_2 - Q_2C_2 - k_2V_2C_2 + E'\left(C_1 - C_2\right)$$

EMBAYMENT MODEL (Cont)

❖ Estimation of Diffusion

Conservatve are substances that do not have a decomposition process. (k=0)

Mass balance for the bay(2):

$$V_2 \frac{dC_2}{dt} = W_2 - Q_2 C_2 + E'\left(C_1 - C_2\right)$$

Steady – state slotions

$$E' = \frac{W_2 - Q_2 C_2}{C_2 - C_1}$$

Example 2: For a lake and bay, parameters presented in the table

	BAY		LAKE		
Volume	V_2	8	V_1	3507	10^9 m^3
Depth	H_2	5.81	H_1	60.3	m
Surface area	A_2	1,376	A_1	58,194	10^6 m^2
Outflow	Q_2	7	Q_1	161	10^9 m^3yr^{-1}
Chloride concentration	C_2	15.2	C_1	5.4	g m^{-3}
Chloride loading	W_2	0.353	W_1	0	10^{12} g yr^{-1}
Phosphours loading	W_{p2}	1.42	W_{p1}	4.05	10^{12} mg yr^{-1}

Note that the mass diffusion coefficient is determined based on the concentration gradient of Chlo (preservative). Determine the diffusion coefficient and mass-transfer coefficient for the process.

Solutions:

$$E' = \frac{0.353 \times 10^{12} - [7 \times 10^9 \times 15.2]}{15.2 - 5.4} = 25.2 \times 10^9 \quad m^3/năm$$

Mass- transfer coefficient:

$$v_d = \frac{E'}{A_c} = \frac{25.2 \times 10^9}{0.17 \times 10^6} = 1.48 \times 10^5 \, m/năm$$

Diffusion coefficient

$$E = v_d l = 1.48 \times 10^5 \times 10 \times 10^3 = 1.48 \times 10^9 \, m^2/năm$$

<u>Example 3:</u> For a lake and bay, parameters presented in the table

	BAY		LAKE		
Volume	V_2	8	V_1	3507	10^9 m^3
Depth	H_2	5.81	H_1	60.3	m
Surface area	A_2	1,376	A_1	58,194	10^6 m^2
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Chloride loading	W_2	0.353	W_1	0	10^{12} g yr^{-1}
Phosphours loading	W_{p2}	1.42	W_{p1}	4.05	10^{12} mg yr^{-1}

For the deposition rate of phosphorus is v = 16 m/yr. Determine (a) the concentration of inflows, (b) the concentration in the stable state of the lake and bay.

Solution

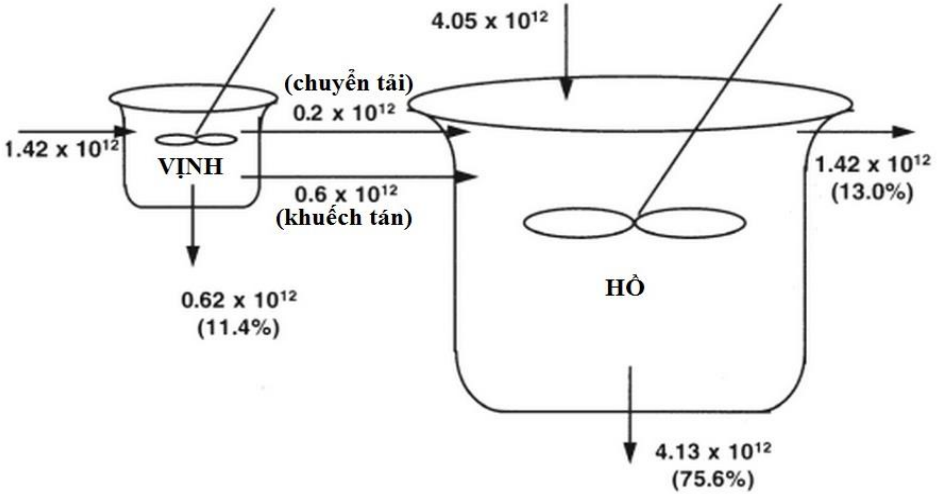
(a) Inflow flow

$$Q_{1,in} = Q_1 - Q_2 = 161\times10^9 - 7\times10^9 = 154\times10^9\, m^3 yr^{-1}$$

Inflow concentration

$$C_{1,in} = \frac{4.05\times10^{12}}{154\times10^9} = 26.3\mu g L^{-1}$$

$$C_{2,in} = \frac{1.42\times10^{12}}{7\times10^9} = 202.9\mu g L^{-1}$$



(b) Steady – state solution

$$C_1 = \frac{1}{1.102\times10^{12}}W_{p1} + \frac{1}{1.857\times10^{12}}W_{p2} = 3.671 + 0.768 = 4.44\mu g L^{-1}$$

$$C_2 = \frac{1}{2.373\times10^{12}}W_{p1} + \frac{1}{5.345\times10^{12}}W_{p2} = 1.705 + 26.658 = 28.36\mu g L^{-1}$$

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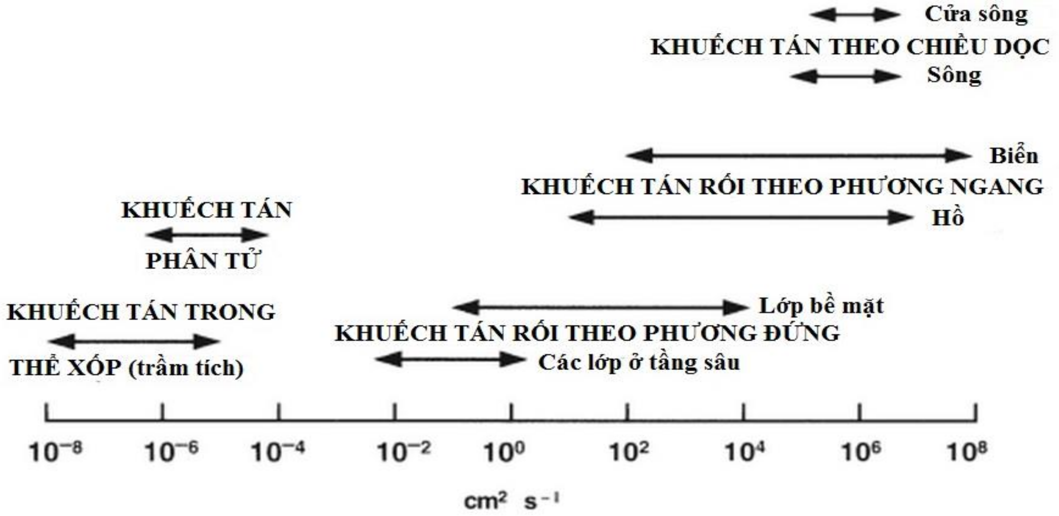
ADDITIONAL TRANSPORT MECHANISMS

❖ Diffusion

Mass is spreads via random molecular motion or due to large-scale agitation in the water environment.

- There are two important differences.:
  - + Turbulent motion is greater than the random molecular motion, therefore mixing caused by tangle diffusion is greater than molecular diffusion.
  - + Molecular diffusion takes place uniformly at scale, while Turbulent diffusion takes place in wider-sized ranges. Therefore, diffusion depends on space.

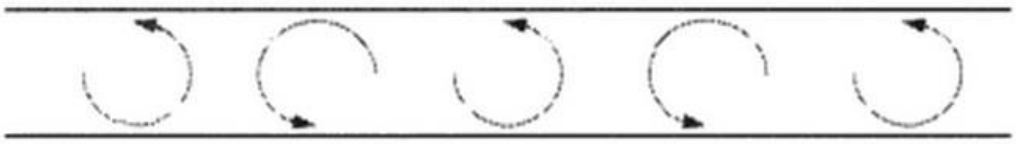


The value range of diffusion coefficients in water and sediment environments

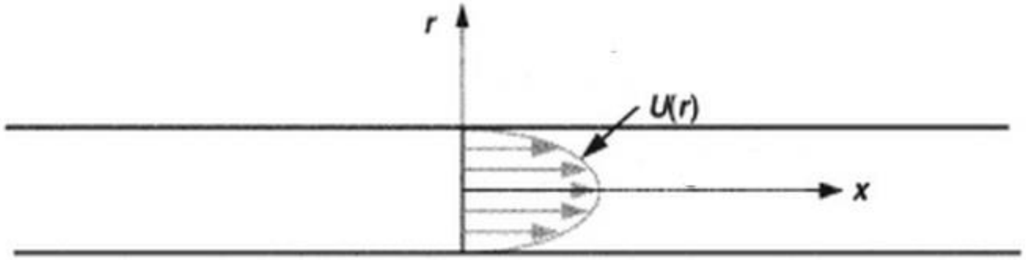
ADDITIONAL TRANSPORT MECHANISMS (cont)

❖ Dispersion

Dispersion involves the process of causing pollutants to spread out. This process is the result of a change in the flow velocity through space.



(a) khuếch tán



(b) phân tán

ADDITIONAL TRANSPORT MECHANISMS (cont)

CONDUCTION AND CONVECTION

Conduction and convection are two pro- cesses that originate from heat transfer and aerodynamics that are roughly analogous to diffusion and advection

- Conduction refers to the transfer of heat by molecular activity from one sub- stance to another or through a substance
- **Convection** which generally refers to the motions in a fluid that result in the transport and mixing of the fluid's properties, takes two forms.
  - Free convection refers to vertical atmospheric motions due to the buoyancy of heated or cooled fluid, For example in meteorology, the rising of heated surface air and sinking of cooler air aloft is called "free convection
  - Forced convection is due to external forces. An example is the lateral movement of heat or mass due to the wind. Thus forced convection is akin to advection

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LECTURE

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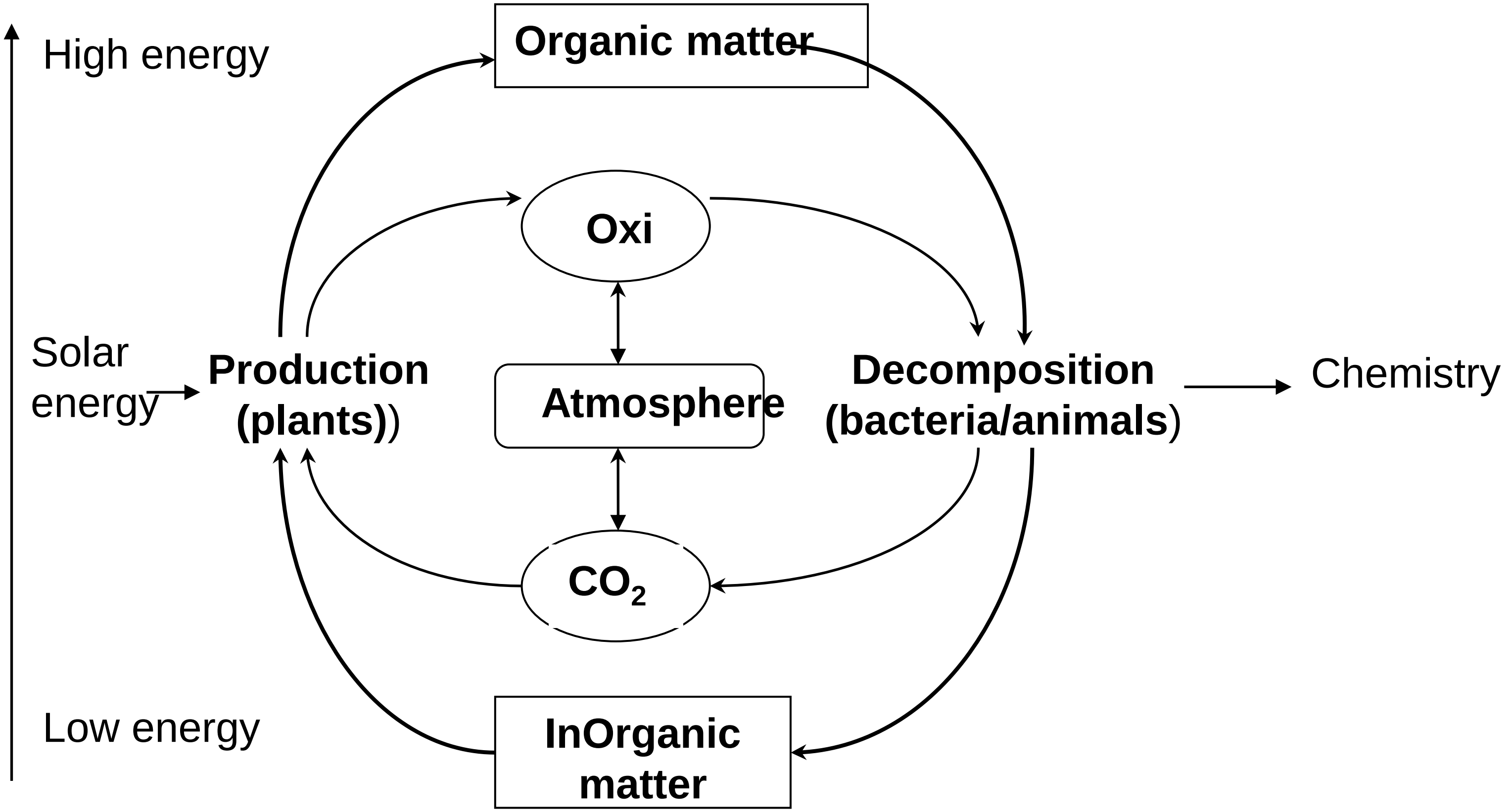
Lecture 8

BOD AND OXYGEN SATURATION

Lecturer: Prof. Nguyen Ky Phung
MSc. Dang Thi Thanh Le

Contents

ORGANIC MATTER DECOMPOSITION AND PRODUCTION CYCLE



High energy

Solar energy

Low energy

Organic matter

Oxi

Production (plants))

Atmosphere

Decomposition (bacteria/animals)

Chemistry

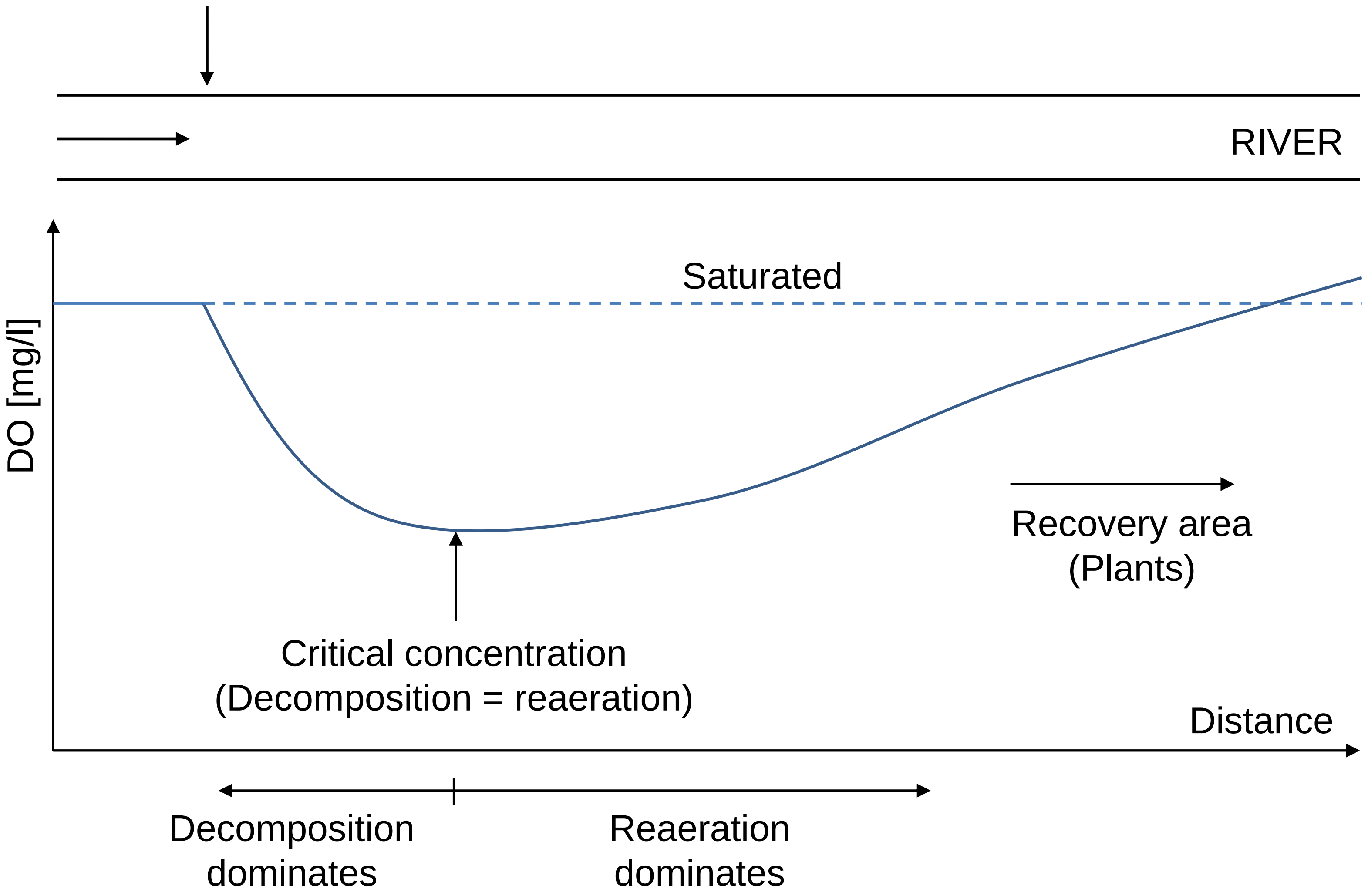
CO_2

InOrganic matter

The cycle above can be described by chemical equations:

$$6CO_2 + 6H_2O \underset{\text{respiration}}{\overset{\text{photosynthesis}}{\rightleftharpoons}} C_6H_{12}O_6 + 6O_2$$

SATURATED DISSOLVED OXYGEN



EXPERIMENT

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The organic matter decomposition cycle is described by the following chemical equation:

$$C_6H_{12}O_6 + 6O_2 \xrightarrow{\text{Respiratory}} 6CO_2 + 6H_2O$$

Mass balancing equation:

Glucose: $$V\frac{dg}{dt} = -k_1 Vg$$

Oxy: $$V\frac{do}{dt} = -r_{og}k_1 Vg$$

k_1: rate of decomposition [1/day]

r_{og}: the stoichiometric of oxygen consumed to glucose decomposition (mg O/mg-glucose)

At t = 0: $g = g_0$ và $o = o_0$

$$g = g_0 e^{-k_1 t}$$

$$o = o_0 - r_{og}g_0\left(1 - e^{-k_1 t}\right)$$

EXPERIMENT (TT)

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Example 1: Put 2mg of glucose in a 250ml bottle, then add a small amount of bacteria. Fill the bottle with water and close the lid. The initial oxygen level was 10mgL^{-1}. If glucose breaks down at a rate of 0.1/day, determine the concentration of oxygen as a function over time in this experiment.

EXPERIMENT (cont)

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Solution:

Initial concentration of glucose

$$g_0 = \frac{2\ \mathrm{mg}}{250\ \mathrm{mL}}\left(\frac{1000\ \mathrm{mL}}{\mathrm{L}}\right) = 8\mathrm{mg/L}$$

stoichiometry:

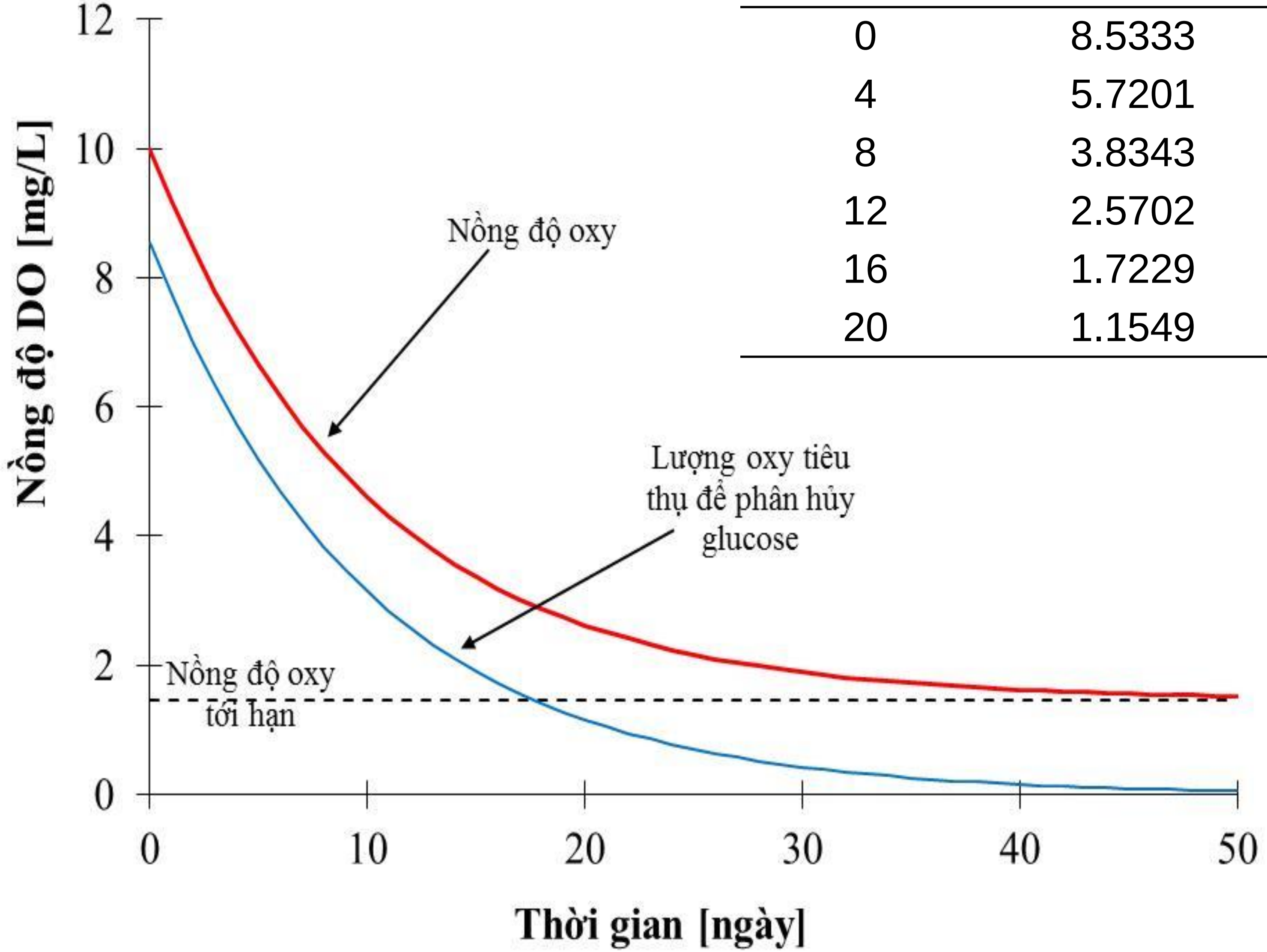
$$r_{og} = \frac{6(32)}{6\times 12 + 1\times 12 + 6\times 16} = 1.0667\ \mathrm{mgO/mg\ glucose}$$

Glucose levels break down and oxygen depletion over time:

$$g = 8e^{-0.1t}$$

$$o = 10 - 8.5333(1 - e^{-0.1t})$$

Time (day)	Glucose (mgO/L	Oxy (mgO/L)
0	8.5333	10.000
4	5.7201	7.187
8	3.8343	5.301
12	2.5702	4.037
16	1.7229	3.190
20	1.1549	2.622



BIOCHEMICAL OXYGEN DEMAND (BOD)

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- ❖ **Biochemical oxygen demand (BOD)** is the amount of oxygen needed for microorganisms to oxidize organic matter over a certain period of time.
- ❖ Mass balancing equation for oxidizing organic matter in the bottle:
- ❖ $V\frac{dL}{dt} = -k_1 VL$

L: the amount of oxidized organic matter left in the bottle: [mgO L^{-1}]

At t = 0: L = L_0 Experiment with the equation now.

$$L = L_0 e^{-k_1 t}$$

Amount of oxygen consumed during decomposition:

$$y = L_0 - L \rightarrow y = L_0\left(1 - e^{-k_1 t}\right)$$

Where y is BOD (mgO L^{-1}); L_0 is the initial concentration of oxidized organic matter (expressed in units of oxygen) or ultimate BOD

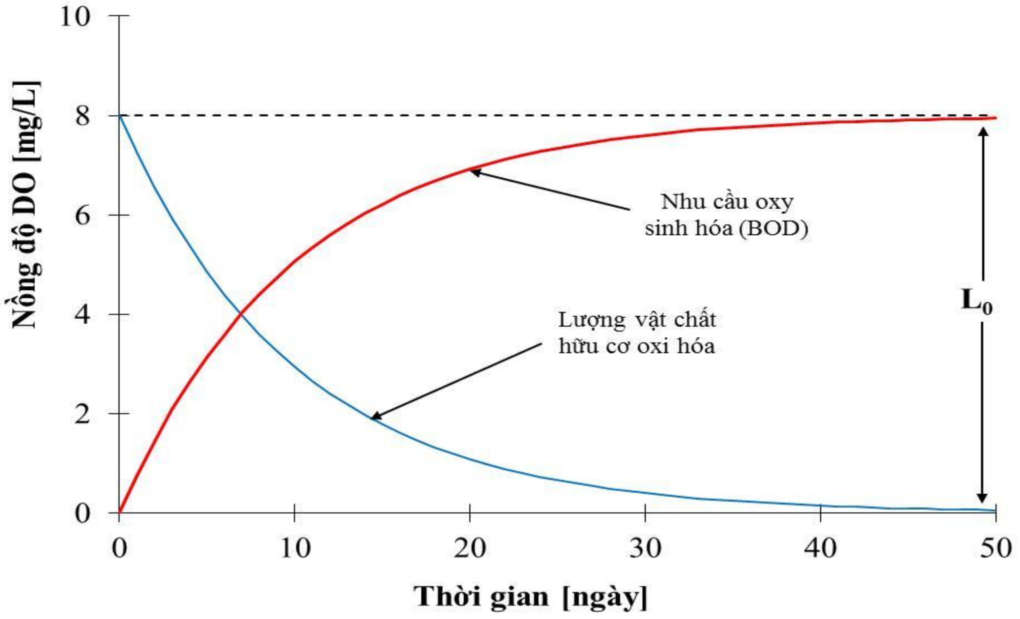
BIOCHEMICAL OXYGEN DEMAND (BOD)

Mass balance equation for oxygen:

$$V\frac{do}{dt} = -k_1 V L_0 e^{-k_1 t}$$

At t = 0: $o = o_0$, the solution of the equation

$$o = o_0 - L_0\left(1 - e^{-k_1 t}\right)$$



The L0 value is the initial concentration of dissolved organic matter or final BOD

BOD FOR A STREAM

Mass balance equation for a Stream

$$\frac{\mathrm{dL}}{\mathrm{dt}} = -\mathrm{U}\frac{\mathrm{dL}}{\mathrm{dx}} - \mathrm{k_r L}$$

k_r = total removal rate (d^{-1}) includes the process of decomposition and sttling,

$$\mathrm{k_r} = \mathrm{k_d} + \mathrm{k_s}$$

In a steady - state

$$0 = -\mathrm{U}\frac{\mathrm{dL}}{\mathrm{dx}} - \mathrm{k_r L}$$

Let's say at the point of discharge, the mixing process is complete.

$$\mathrm{L_0} = \frac{\mathrm{Q_w L_w + Q_r L_r}}{\mathrm{Q_w + Q_r}}$$

Tại t = 0: L = L_0 the solution as below

$$\mathrm{L} = \mathrm{L_0 e}^{-\frac{k_r}{U}x}$$

BOD FOR A STREAM

Exercises

A discharge stream (flow 2 cms, BOD is 10 mg/l) from an active sludge treatment plant poured into a river (flow 5 cms and BOD is 0 mg/l). Characteristics of the river: $k_{r,}$ 20 = 0.2 days-1, cross-section area is 25 m^2, and T = 28°C, θ =1,047 for BOD decomposition.

a. Determination of BOD levels at the mixing point

b. How far is the distance from the discharge point so that the BOD concentration reaches 5% of the original BOD valu?

DO SATURATION

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- **The effect of temperature.** (APHA 1992)

$$\ln o_{sf} = -139.34411 + \frac{1.575701.10^{5}}{T_a} - \frac{6.642308.10^{7}}{T_a^2} + \frac{1.243800.10^{10}}{T_a^3} - \frac{8.621949.10^{11}}{T_a^4}$$

o_{sf} Is the concentration of do saturation at the pressure 1 atm (mg L^{-1}) and T_a is absolute temperature. (K), $T_a = T + 273.15$

- **Effects of salinity** (APHA 1992)

$$\ln o_{ss} = \ln o_{sf} - S\left(1.7674.10^{-2} - \frac{1.0754.10^{1}}{T_a} + \frac{2.1407.10^{3}}{T_a^2}\right)$$

o_{ss} is the concentration of do saturation in seawater at pressure 1 atm (mgL^{-1}), S is salinity (gL^{-1} or ppt).

S=1.80655 x Chlor

Chlor is a concentration of chloride (ppt). The higher the salinity, the lower the amount of oxygen in the water.

DO SATURATION (cont)

- **Effects of pressure** (APHA 1992)

$$o_{sp} = o_{si}\, p \left[\frac{\left(1\text{-}\frac{\mathrm{p}_{\mathrm{wv}\text{-}}}{p}\right)\left(1-\theta p\right)}{\left(1-p_{wv}\right)\left(1-\theta\right)} \right]$$

p = atmospheric pressure (atm);

o_{sp} = saturated oxygen concentration at pressure p (mg L^{-1});

o_{s1} = saturation concentration of dissolved oxygen at pressure 1 atm (mgL^{-1});

p_{wv} = partial pressure part of steam (atm)

$$\ln p_{wv} = 11.8571 - \frac{3840.70}{T_a} - \frac{216.961}{T_a^2} \qquad \theta = 0.000975 - 1.426.10^{-5} T + 6.436.10^{-8} T^2$$

Zison el at (1978) advanced approximation formula based on altitude

$$o_{sp} = o_{s1}\left[1 - 0.1148 * elev(km)\right]$$

DO SATURATION (cont)

Exercises

DO, temperature, and salinity are measured at an estuaries as follows:

Distance from the sea	30	20	10
Temperature, °C	25	22	18
Salinity, ppt	5	10	20
DO	5	6.5	7.5

Calculate the percentage of saturated oxygen at the above 3 locations

MARE

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LECTURE

MODELLING THE MARINE ENVIRONMENT

Lecturer: Prof. Nguyen Ky Phung
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Lecture 9

STRETER- PHELPS MODEL

Lecturer: Prof. Nguyen Ky Phung
MSc. Dang Thi Thanh Le

CONTENTS

- **Point source**
- **Distribution source**

STRETER- PHELPS
POINT SOURCES

EXPERIMENT

Mass balance equation:

BOD: $$V\frac{dL}{dt} = -k_d VL$$

DO: $$V\frac{do}{dt} = -k_d VL + k_a V(o_s - o)$$

DO Deficit: $D = o_s - o$ $\longrightarrow \frac{dD}{dt} = -\frac{do}{dt}$

$$\longrightarrow \quad V\frac{dD}{dt} = k_d VL - k_a VD$$

If at = 0: L = L_o and D = O, above equation can be differentiated

$$L = L_0 e^{-k_d t}$$

$$D = \frac{k_d L_0}{k_a - k_d}\left(e^{-k_d t} - e^{-k_a t}\right)$$

EXPERIMENT (Cont)

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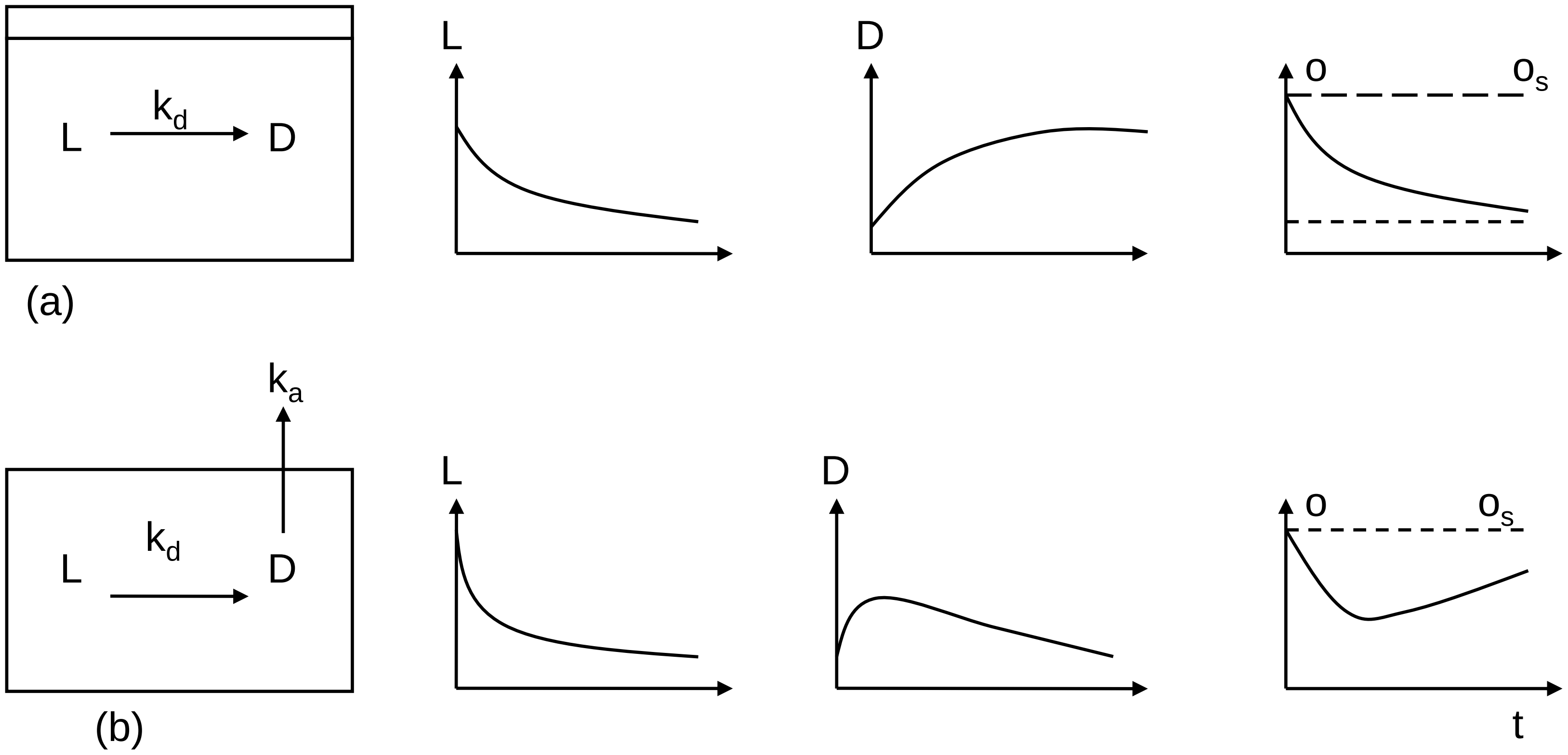
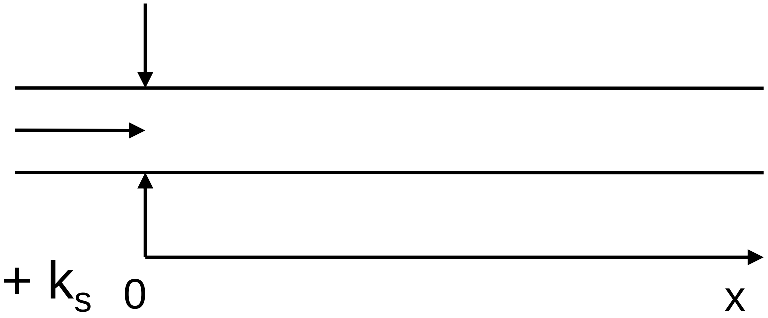


BOD decomposition process in (a) closed and (b) open system

POINTS SOURCE -STREETER-PHELPS EQUATION

A stream with a single-point source of BOD. The river segment is in a steady state and is characterized by plug flow with constant hydrological and geometry (PFR system). This is the simplest manifestation of the classic Streeter – Pheld model

Mass balance: $0 = -U\frac{dL}{dx} - k_r L$



$$0 = -U\frac{dD}{dx} + k_d L - k_a D$$

với $k_r = k_d + k_s$

If $L = L_o$ và $D = D_o$ at $t = 0$, then these equation can be solved for

$$L = L_0 e^{-\frac{k_r}{U}x}$$

$$D = D_0 e^{-\frac{k_a}{U}x} + \frac{k_d L_0}{k_a - k_r}\left(e^{-\frac{k_r}{U}x} - e^{-\frac{k_a}{U}x}\right)$$

POINTS SOURCE -STREETER-PHELPS EQUATION (cont)

- **Some formulas calculate reaeration coefficient (k_a)**

O'Connor-Dobbins (1956)

$$k_a = 3.93 \frac{U^{0.5}}{H^{1.5}}$$

Churchill (1962)

$$k_a = 5.026 \frac{U}{H^{1.67}}$$

Owens và Gibbs (1964)

$$k_a = 5.32 \frac{U^{0.67}}{H^{1.85}}$$

where: k_a (1/day), U (m/s), H (m)

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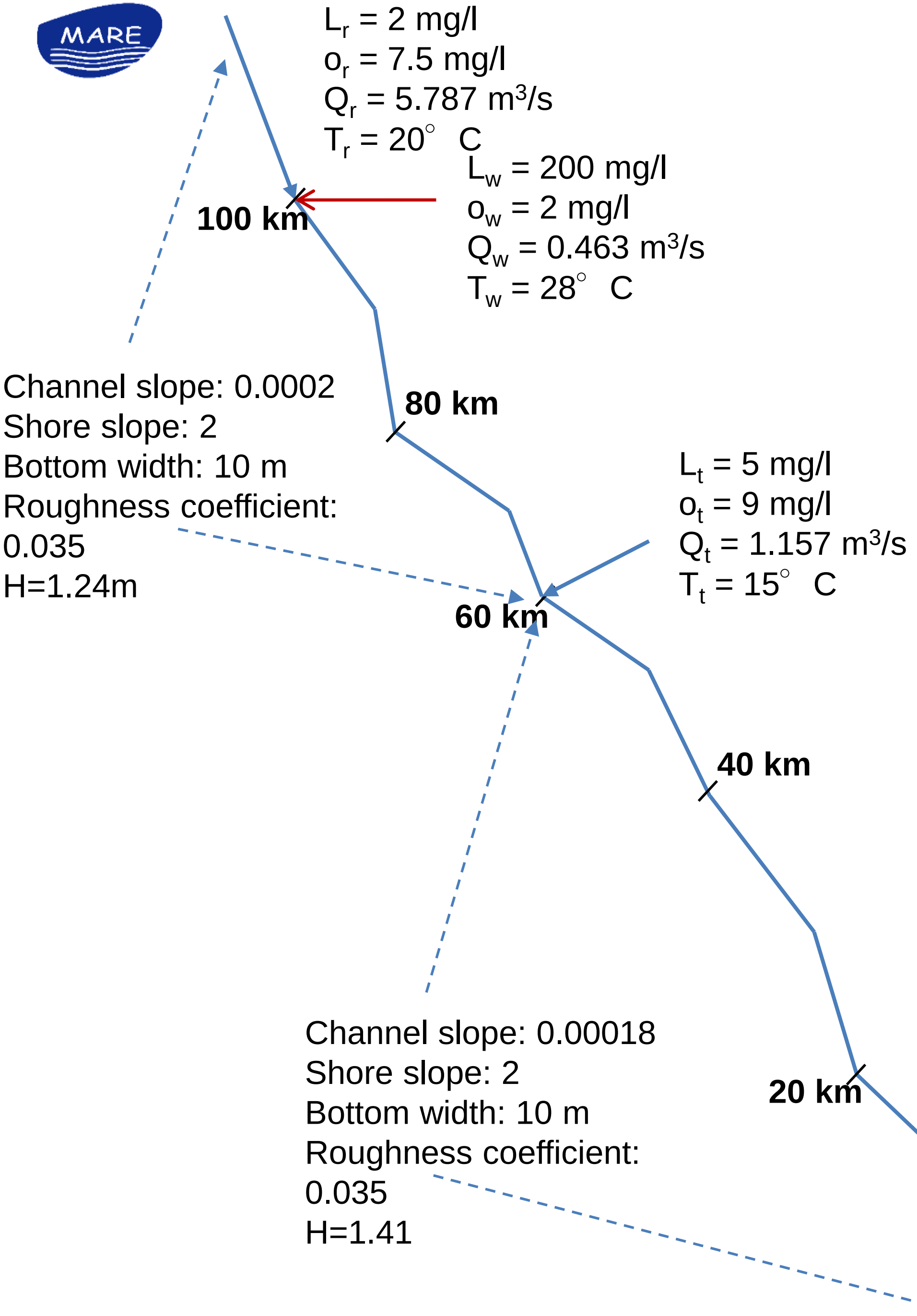
Example 1:

The river is described as shown in the picture. The de-oxygenation rate for CBOD is 0.5 day^{-1} in 20^{o}C. From a distance of 20 km downstream from the plant, the CBOD deposition rate is 0.25 day^{-1}.

Assuming that O'Connor-Dobbins's reaeration formula is used, calculates the concentration of DO in the system.



$L_r = 2$ mg/l
$o_r = 7.5$ mg/l
$Q_r = 5.787$ m^3/s
$T_r = 20^{\circ}$ C

$L_w = 200$ mg/l
$o_w = 2$ mg/l
$Q_w = 0.463$ m^3/s
$T_w = 28^{\circ}$ C

$L_t = 5$ mg/l
$o_t = 9$ mg/l
$Q_t = 1.157$ m^3/s
$T_t = 15^{\circ}$ C

Channel slope: 0.0002
Shore slope: 2
Bottom width: 10 m
Roughness coefficient: 0.035
H=1.24m

Channel slope: 0.00018
Shore slope: 2
Bottom width: 10 m
Roughness coefficient: 0.035
H=1.41

The hydrographic morphological parameters and reaction kinetics of the system

Parameter	Units	KP > 100	KP: 100-60	KP < 60
Depth	m	1.19	1.24	1.41
Area	m^2	14.71	15.5	18.05
Flow	$m^3\ s^{-1}$	5.787	6.250	7.407
	$m^3\ d^{-1}$	500,000	540,000	640,000
Velocity	$m\ s^{-1}$	0.393	0.403	0.410
	$m\ d^{-1}$	33,955	34,891	35,524

Parameter	KP > 100	KP: 100 - 80	KP: 80 - 60	KP < 60
T (^oC)	20	20.59	20.59	19.72
O_s (mgL^{-1})	9.092	8.987	8.987	9.143
k_a (d^{-1})	1.902	1.842	1.842	1.494
k_r (d^{-1})	0.50	0.764	0.514	0.494
k_d (d^{-1})	0.50	0.514	0.514	0.494

ESTUARY STREETER-PHELPS MODEL

With a diffusion system such as an estuary the Streeter-Phelps equation can be written as:

$$0 = E\frac{d^2L}{dx^2} - U\frac{dL}{dx} - k_r L$$

$$0 = E\frac{d^2D}{dx^2} - U\frac{dD}{dx} + k_d L - k_a D$$

If $L = L_o$ và $D = D_o$ at $t = 0$, the solution for BOD is

$$L = L_0 e^{j_1 r x} \qquad x \leq 0$$

$$L = L_0 e^{j_2 r x} \qquad x \geq 0$$

ESTUARY STREETER-PHELPS MODEL (cont)

The solution to oxygen deficiency

$$D = \frac{k_d}{k_a - k_r} \frac{W}{Q} \left(\frac{e^{j_{1r}x}}{\alpha_r} - \frac{e^{j_{1a}x}}{\alpha_a} \right) \qquad x \leq 0$$

$$D = \frac{k_d}{k_a - k_r} \frac{W}{Q} \left(\frac{e^{j_{2r}x}}{\alpha_r} - \frac{e^{j_{2a}x}}{\alpha_a} \right) \qquad x \geq 0$$

where

$$L_0 = \frac{W}{\alpha_r Q}$$

$$\alpha_r = \sqrt{1 + \frac{4k_r E}{U^2}}$$

$$\alpha_a = \sqrt{1 + \frac{4k_a E}{U^2}}$$

$$\begin{matrix} j_{1r} \\ j_{2r} \end{matrix} = \frac{U}{2E}\left(1 \pm \alpha_r\right)$$

$$\begin{matrix} j_{1a} \\ j_{2a} \end{matrix} = \frac{U}{2E}\left(1 \pm \alpha_a\right)$$

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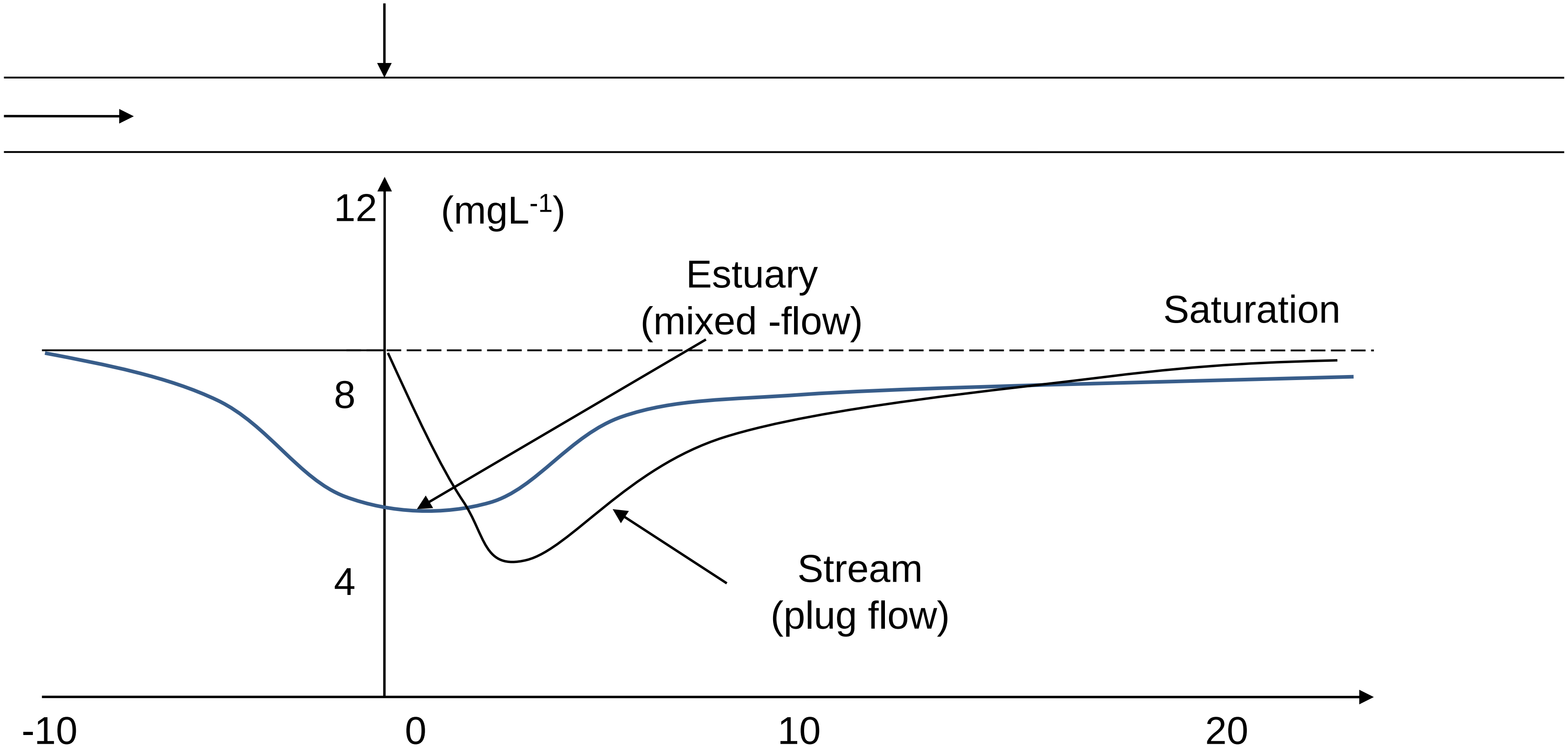
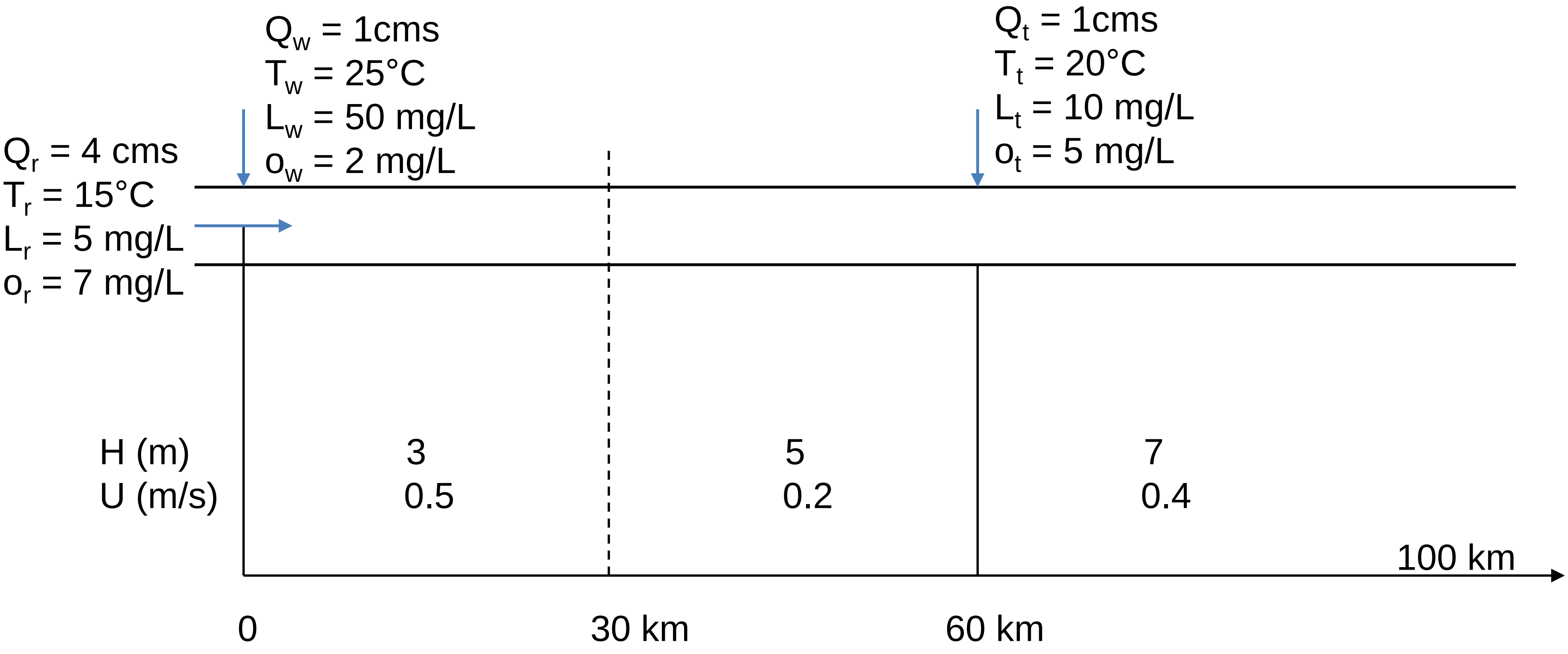
ESTUARY STREETER-PHELPS MODEL (cont)



DO chart for a point waste source into an estuary

Exercises 1: Determine BOD and DO for the following section of the river.

For the decomposition rate of BOD at 20°C is 0.35 (1/day), θ = 1.047.



Q_w = 1cms
T_w = 25°C
L_w = 50 mg/L
o_w = 2 mg/L

Q_t = 1cms
T_t = 20°C
L_t = 10 mg/L
o_t = 5 mg/L

Q_r = 4 cms
T_r = 15°C
L_r = 5 mg/L
o_r = 7 mg/L

H (m)	3	5	7
U (m/s)	0.5	0.2	0.4

0
30 km
60 km
100 km

STRETER- PHELPS DISSTRIBUTED SOURCES

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PARAMETERIZATION OF DISTRIBUTION SOURCES

Mass balance in case of plug- flow:

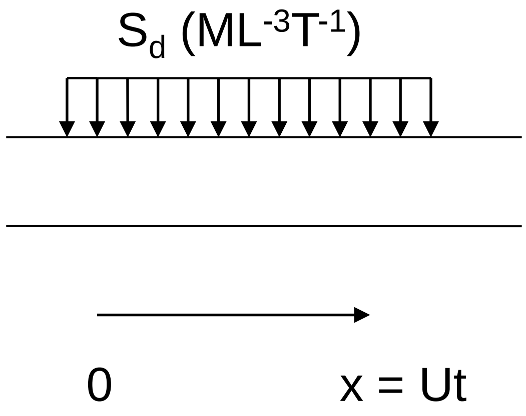
$$0 = -U\frac{dc}{dx} - kc + S_d$$

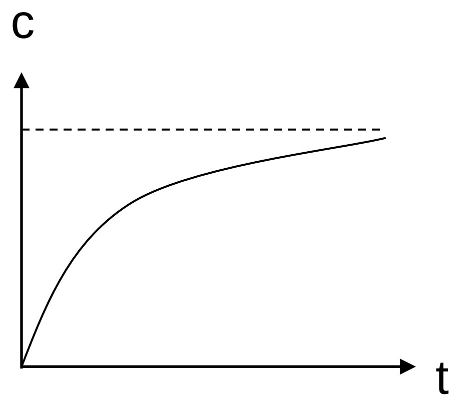
If at = 0: c = c_o , then the solution is:

$$c = \frac{S_d}{k}\left(1 - e^{-kt}\right)$$

S_d = the rate of the distribution source. ($ML^{-3}T^{-1}$).

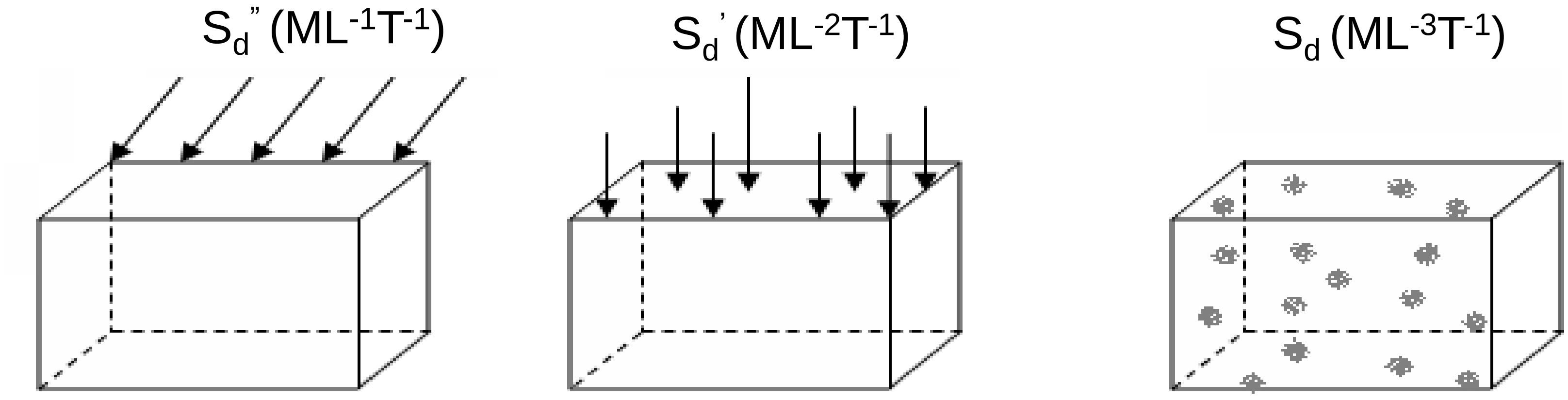
t = travel time , t = x/U.





PARAMETERIZATION OF DISTRIBUTION SOURCES (cont)

❖ Distributed source types



(a) Line load

(b) Area load

(c) Volume load

$$S_d = S_d'' \frac{L}{V} = \frac{S_d''}{A_c}$$

$$S_d = S_d' \frac{A_s}{V} = \frac{S_d'}{H}$$

L = distribution source length

V = volume

A_c = the cross-section area of the segment in which the source is included.

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BOD DISTRIBUTION SOURCE

- Source: bottom sludge with high organic concentration

Mass balance:

$$0 = -U\frac{dL}{dx} - k_r L + S_L$$

Where S_L = the speed of the BOD distribution source(gm^{-3}d^{-1})

k_r = rate of BOD removal (day^{-1}).

Solution when t = 0: L = 0:

$$L = \frac{S_L}{k_r}\left(1 - e^{-k_r t}\right)$$

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BOD DISTRIBUTION SOURCE (cont)

The mass balance for oxygen deficiency is written as follows

$$0 = -U\frac{dD}{dx} - k_a D + \frac{k_d}{k_r} S_L \left(1 - e^{-k_r t}\right)$$

At t = 0: D = 0, The solution to mass balance is as follows:

$$D = \frac{k_d S_L}{k_r k_a}\left(1 - e^{-k_a t}\right) - \frac{k_d S_L}{k_r\left(k_a - k_r\right)}\left(e^{-k_r t} - e^{-k_a t}\right)$$

Where t = travel time. (t = x/U)

DO DISTRIBUTION SOURCE

- Source: Plants (photosynthesis)
- Deposition: plants (respiration), SOD
- Mass balance

$$0 = -U\frac{dD}{dx} - k_a D - P + R + \frac{S_B^{'}}{H}$$

Where P, R: the rate of photosynthesis and respiration of plants (gm^{-3}nday^{-1})

$S_B^{'}$ = oxygen demand rate for sediment (gm^{-2}day^{-1})

H = depth (m)

- The solution when t = 0: L = 0:

$$D = \frac{-P + R + \left(S_B^{'}/H\right)}{k_a}\left(1 - e^{-k_a t}\right)$$

TOTAL STREETER-PHELPS MODEL

- General solution to the points and distribution sources of BDO and DO.
- $$L = \underbrace{\boxed{L_0 e^{-k_r t}}}_{\text{Point source}} + \underbrace{\boxed{\frac{S_L}{k_r}\left(1 - e^{-k_r t}\right)}}_{\text{Distribution source}}$$

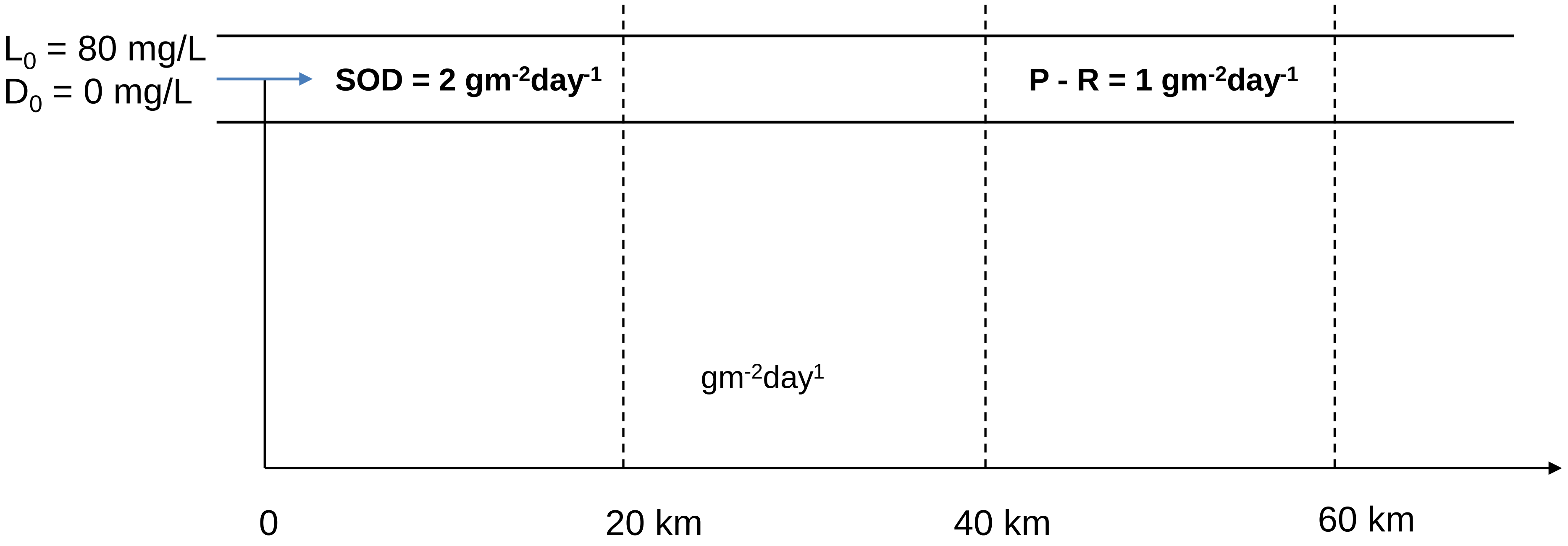
$$D = \overset{\text{Point deficit}}{\boxed{D_0 e^{-k_r t}}} + \overset{\text{Point BOD}}{\boxed{\frac{k_d L_0}{k_a - k_r}\left(e^{-k_r t} - e^{-k_a t}\right)}}$$

$$\boxed{+ \frac{-P + R + \left(S_B^{'} / H\right)}{k_a}\left(1 - e^{-k_a t}\right)} \longleftarrow \text{Distribution Deficit}$$

$$\boxed{+ \frac{k_d S_L}{k_r k_a}\left(1 - e^{-k_a t}\right) - \frac{k_d S_L}{k_r\left(k_a - k_r\right)}\left(e^{-k_r t} - e^{-k_a t}\right)} \longleftarrow \text{Distributed BOD}$$

Exercises:

Determine bod and do for the following section of the river.



L_0 = 80 mg/L
D_0 = 0 mg/L

SOD = 2 $gm^{-2}day^{-1}$

P - R = 1 $gm^{-2}day^{-1}$

$gm^{-2}day^{1}$

0

20 km

40 km

60 km

U (mps)	0.1	0.15	0.1
H (m)	0.8	1	1
k_r (ngày^{-1})	0.2	0.1	0.1
k_d (ngày^{-1})	0.1	0.1	0.1
k_a (ngày^{-1})	1	1.2	1.2
o_s (mgL^{-1})	10	9	8